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The Neveu-Schwarz and Ramond algebras of logarithmic superconformal field theory

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Abstract: We describe the general features of the Neveu-Schwarz and Ramond sectors of logarithmic conformal field theories with $N = 1$ supersymmetry. Three particular systems are examined in some detail — D-brane recoil, a superconformal extension of the $c = -2$ model, and the supersymmetric $\mathrm{SU}(2)_2$ WZW model.

Keywords: Conformal Field Models in String Theory, Supersymmetry and Duality.

Contents

1. Introduction and summary

Logarithmic conformal field theories [1, 2] have recently been attracting a lot of attention because of their diverse range of applications, from condensed matter models of disorder [3, 4] to applications involving gravitational dressing of two-dimensional field theories [5], a general analysis of target space symmetries in string theory [6], D-brane recoil [7, 8], and AdS backgrounds in string theory and also M-theory [9] (see [10] for reviews and more exhaustive lists of references). They lie on the border between conformally invariant and general renormalizable field theories in two dimensions. A logarithmic conformal field theory is characterized by the property that its correlation functions differ from the standard conformal field theoretic ones by terms which contain logarithmic branch cuts. Nevertheless, it is a limiting case of an ordinary conformal field theory which is still compatible with conformal invariance and which can still be classified to a certain extent by means of conformal data.

The current understanding of logarithmic conformal field theories lacks the depth and generality that characterizes the conventional conformally invariant field theories. Most of the analyses so far pertain to specific models, and usually to those involving free field realizations. Nevertheless, some general properties of logarithmic conformal field theories are now very well understood. For example, an important deviation from standard conformal field theory is the non-diagonalizable spectrum of the Virasoro hamiltonian operator L_0, which connects vectors in a Jordan cell of a certain size. This implies that the logarithmic operators of the theory, whose correlation functions exhibit logarithmic scaling violations, come in pairs, and they appear in the spectrum of a conformal field theory when two primary operators become degenerate. It would be most desirable to develop methods that would classify and analyse the origin of logarithmic singularities in these models in as general a way as possible, and in particular beyond the free field prescriptions. Some modest steps in this direction have been undertaken recently using different approaches. For instance, an algebraic approach is advocated in [11, 12] and used to classify the logarithmic triplet theory as well as certain non-unitary, fractional level Wess-Zumino-Witten (WZW) models. The characteristic features of logarithmic conformal field theories are described within this setting in terms of the representation theory of the Virasoro algebra. An alternative approach to the construction of logarithmic conformal field theories starting from conformally invariant ones is proposed in [13]. In this setting, logarithmic behaviour arises in extended models obtained by appropriately deforming the fields, including the energy-momentum tensor, in the chiral algebra of an ordinary conformal field theory.

From whatever point of view one wishes to look at logarithmic conformal field theory, an important issue concerns the nature of the extensions of these models to include world-sheet supersymmetry. In many applications, most notably in string theory, supersymmetry plays a crucial role in ensuring the overall stability of the target space theory. The purpose of this paper is to analyse in some detail the general characteristics of the $N = 1$ supersymmetric extension of logarithmic conformal field theory. These models were introduced in [14]–[16], where some features of the Neveu-Schwarz (NS) sector of the superconformal algebra were described. In the following we will extend and elaborate on these studies, and

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further incorporate the Ramond (R) sector of the theory. In addition to unveiling some general features of logarithmic superconformal field theories, we shall study in detail how these novel structures emerge in three explicit realizations.

1.1 Outline and summary of results

In the remainder of this section we shall outline the structure of the rest of this paper, and summarize in detail the main accomplishments of the present work section by section.

Section 2: Generalities. We introduce the relations of the $N = 1$ logarithmic superconformal algebra and define its highest weight representations in both the Neveu-Schwarz and Ramond sectors of the worldsheet field theory. We pay particular attention to the structure of the spin operators which connect the Ramond highest weight states with the vacuum representation of the Neveu-Schwarz algebra. We show that, in addition to the standard R sector spin fields of superconformal field theory [17], there are excited spin fields produced by the logarithmic pairs which generate, from their action on the NS vacuum module, representations of an ordinary logarithmic conformal algebra inside the Ramond algebra. Generally, this yields two orthogonal Jordan cells for the action of L_0 on the Ramond highest weight representation. The degeneracy is lifted by projecting onto the sector of unbroken Ramond supersymmetry, leaving a single copy of the logarithmic conformal field theory inside the R algebra, analogously to the NS sector.

Section 3: Correlators. We examine the problem of determining the correlation functions of a generic logarithmic superconformal field theory. We derive the appropriate superconformal Ward identities and present explicit expressions for the two-point and three-point Green's functions of the logarithmic superfields in the NS sector [15, 16]. Using the local monodromy constraints provided by the spin fields, we derive the two-point functions of the logarithmic operators in the R sector. We also derive the two-point functions of all spin fields in the theory, as well as the mixed three-point correlators among the two types of spin operators and the fermionic components of the logarithmic superfields.

Section 4: Singular vectors. We briefly discuss some issues related to null states of the logarithmic superconformal field theory. We show that logarithmic superfields of vanishing superconformal dimension generate a non-trivial, dynamical fermionic symmetry of the theory. We then describe the three possible ways to generate local superconformal algebras involving the spin operators. One way is to project onto the NS sector, another is to project onto the supersymmetric Ramond ground state. A third way is to project the chiral symmetry algebra onto operators of positive fermion parity [17], which leaves a local quantum field theory in which the spin operators together completely determine the entire field content of the logarithmic superconformal field theory.

Section 5: D-brane recoil. We study the first of three explicit examples and revisit, within the context of sections 2–4, the well-studied supersymmetric impulse operators [16] which describe the recoil of D-particles in string theory. We complete the program of constructing quantum states which describe the back-reaction of a D-brane background to

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the scattering of closed string states. We exploit the relative simplicity of this model to give a detailed account of the construction of spin operators, introducing techniques that are further used in subsequent sections. We then explicitly construct the target space fermionic vertex operator in this problem, and thereby complete the construction of supersymmetric recoil states for D-particles in type IIA superstring theory.

Section 6: Symplectic fermions. We develop the superconformal extension of the best understood logarithmically extended $c_{p,q}$ minimal model, the $c = -2$ model, through its local formulation in terms of symplectic fermions [18]. This example is responsible for the onset of logarithmic behaviour in many problems. The superconformal symplectic fermion is formulated using the standard expressions for chiral scalar superfields (with opposite statistics). We construct spin operators and show that, in contrast to the other examples of this paper which each possess a unique state of unbroken Ramond supersymmetry, there is a whole continuum of Ramond ground states labelled by their U(1) charge in the underlying non-compact, fermionic current algebra of the model. They generate infinitely many inequivalent highest-weight representations of the Ramond algebra in this case. This infinity appears to be related to some integrability property of the superconformal $c = -2$ model. We then examine the superconformal versions of some of the logarithmically extended algebras that are normally used to classify the bosonic sector of this theory. We derive the supersymmetric extension of the "logarithmic" $sl(2, \mathbb{R})$ Kac-Moody symmetry [18, 19], and hence show that the supersymmetric triplet model is properly classified in terms of a superconformal enhancement of its underlying W-algebra in the bosonic sector. We also study the fusion algebra in the $\mathbb{Z}_2$-twisted version of the model [11], and derive the fusion rules for the supersymmetric extension which give rise to the appropriate indecomposable representations [11, 20] of the $N = 1$ super Virasoro algebra. This formulates the model in purely algebraic terms, beyond its explicit realization in terms of symplectic fermions.

Section 7: WZW models. We consider a third and final example which takes us beyond the minimal models, the $N = 1$ supersymmetric WZW model based on the group SU(2) at level $k \in \mathbb{Z}_+$. These models constitute the simplest interacting quantum field theories within the present context. By employing current algebra methods and using an appropriate Coulomb gas representation, we begin by recalling how the fermionic fields of the theory decouple such that the bosonic sector of the model is equivalent to the bosonic $su(2)$ Kac-Moody algebra at level $k - 2$. This representation enables the construction of the spin operators that define the Ramond sector of the worldsheet field theory. It further enables a very explicit construction of logarithmic operators in the theory via a special deformation of certain primary states of the quantum field theory. Superficially, we find that these operators exist in the theory for any values of the level number of the form $k = n(n + 1)$, where n is any positive integer. For these values of k the deformation is exactly marginal, and hence produces an isomorphic superconformal algebra. Furthermore, for these levels we can use the fermionic screening operator of the particular Coulomb gas representation to define a deformation of the underlying chiral symmetry algebra of the theory, along the lines of [13], and hence recover the logarithmic states through a precise extension of the Virasoro algebra. Through the supersymmetric formalism with the given special free field realization of the model, we are thereby able to exhibit, for the first time, explicitly the logarithmic operators for this class of models. Furthermore, our construction, when compared with other models, hints at the fact that logarithmic operators may be generically present in ordinary conformal field theories through "hidden" deformations and extensions, as also suggested in [13].

Although at present we have found no obstruction to the construction of these operators at the stated level numbers, we are only able to rigorously prove that they yield a bonafide logarithmic pair for the very special value $k = 2$. This is the level at which logarithmic behaviour has been previously observed in these models [6, 14]. In fact, within the present formalism we can shed some new light on the special properties of the supersymmetric $\mathrm{SU}(2)_2$ model and its logarithmic behaviour. Let us summarize our findings for the special characteristics of $k = 2$ and contrast them to the cases of affine $su(2)_k$ algebras with $k > 2$:

- The supersymmetric $su(2)_2$ Kac-Moody algebra is determined by a single scalar superfield, and as a consequence it has a unique ground state of unbroken Ramond supersymmetry, as anticipated from the general arguments of section 2. For $k > 2$ the affine algebra is described by means of three scalar superfields and the Ramond supersymmetry is broken.

- For $k = 2$ the logarithmic operators so constructed can be shown to naturally arise through the fusion products of spin $1/2$ primary fields, as they are expected to [6, 14]. For $k > 2$ it is not clear which fusion relations they are associated to, if any.

- At $k = 2$ we can naturally embed the bosonic sector of the WZW model into a twisted $N = 2$ superconformal algebra [21] such that the fermionic screening charge of the WZW model becomes the BRST operator of the Felder complexes of Fock space representations of the $c = 0$ Virasoro algebra. Using the deformation provided by the screening, we are then able to explicitly construct indecomposable representations in which L_0 acquires Jordan blocks [13]. For $k > 2$ this construction does not work.

- For $k = 2$ additional generators of a $c = -2$ Virasoro algebra appear in the spectrum of the quantum field theory. These operators do not generate the logarithmic behaviour associated with the deformation, but rather they dress the logarithmic operators to ensure that the deformations are marginal and hence that an isomorphic superconformal algebra is generated. Instead, through a similar deformation process [13], the $c = -2$ operators generate an independent logarithmic sector of the supersymmetric $\mathrm{SU}(2)_2$ WZW model. For $k > 2$, one may use a Wakimoto representation [22] of the current algebra to construct $c = -2$ sectors of the WZW model for generic values of the level k [23]. Logarithmic structures then emerge again through appropriate deformations of the underlying chiral algebras. However, these deformations do not seem to preserve the superconformal algebra.

We conclude with a comparison of the $su(2)_k$ affine algebras and the $sl(2,\mathbb{R})_k$ WZW models, in which logarithmic operators are known to emerge for any k [13, 24, 25], and also briefly discuss some extensions and applications to coset current algebras.

2. Definition of the $N = 1$ logarithmic superconformal algebra

We will start by looking at an abstract logarithmic superconformal field theory to see what some of the general features are. Throughout we will deal for simplicity with situations in which the two-dimensional field theory contains only a single Jordan cell of rank 2, but our considerations easily extend to more general situations. In this section we shall begin by discussing how to properly incorporate the Ramond sector of the theory.

2.1 Operator product expansions

Consider a logarithmic superconformal field theory defined on the complex plane $\mathbb{C}$ (or the Riemann sphere $\mathbb{C}\cup\{\infty\}$) with coordinate z. For the most part we will only write formulas explicitly for the holomorphic sector of the two-dimensional field theory. We will also use a superspace notation, with complex supercoordinates $\mathsf{z} = (z, \theta)$, where θ is a complex Grassmann variable, $\theta^2 = 0$. The superconformal algebra is generated by the holomorphic super energy-momentum tensor

$$\mathsf{T}(\mathsf{z}) = G(z) + \theta\, T(z) \tag{2.1}$$

which is a chiral superfield of dimension 3/2. Here $T(z)$ is the bosonic energy-momentum tensor of conformal dimension 2, while $G(z)$ is the fermionic supercurrent of dimension 3/2 with the boundary conditions

$$G(\mathrm{e}^{2\pi\,\mathrm{i}}\, z) = \mathrm{e}^{\pi\,\mathrm{i}\,\lambda}\, G(z)\,, \tag{2.2}$$

where $\lambda = 0$ in the NS sector of the theory (corresponding to periodic boundary conditions on the fermion fields) and $\lambda = 1$ in the R sector (corresponding to anti-periodic boundary conditions).

The $N = 1$ superconformal algebra may then be characterized by the anomalous operator product expansion

$$\mathsf{T}(\mathsf{z}_1)\,\mathsf{T}(\mathsf{z}_2) = \frac{\hat{c}}{4}\,\frac{1}{(\mathsf{z}_{12})^3} + \frac{2\theta_{12}}{(\mathsf{z}_{12})^2}\,\mathsf{T}(\mathsf{z}_2) + \frac{1}{2}\,\frac{1}{\mathsf{z}_{12}}\,\mathcal{D}_{\mathsf{z}_2}\mathsf{T}(\mathsf{z}_2) + \frac{\theta_{12}}{\mathsf{z}_{12}}\,\partial_{z_2}\mathsf{T}(\mathsf{z}_2) + \cdots\,, \tag{2.3}$$

where in general we introduce the variables

$$\mathsf{z}_{ij} = z_i - z_j - \theta_i\theta_j\,, \qquad \theta_{ij} = \theta_i - \theta_j \tag{2.4}$$

corresponding to any set of holomorphic superspace coordinates $\mathsf{z}_i = (z_i, \theta_i)$. Here

$$\mathcal{D}_{\mathsf{z}} = \partial_\theta + \theta\,\partial_z\,, \qquad \mathcal{D}_{\mathsf{z}}^2 = \partial_z \tag{2.5}$$

is the superspace covariant derivative, and $\hat{c} = 2c/3$ is the superconformal central charge with c the ordinary Virasoro central charge. An ellipsis will always denote terms which are regular in the operator product expansion as $\mathsf{z}_1 \to \mathsf{z}_2$. By introducing the usual mode expansions

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{\infty} L_n\; z^{-n-2}\,, \\ G(z) &= \sum_{n=-\infty}^{\infty} \frac{1}{2}\,G_{n+(1-\lambda)/2}\; z^{-n-2+\lambda/2} \end{aligned} \tag{2.6}$$

with $L_n^\dagger = L_{-n}$ and $G_r^\dagger = G_{-r}$, the operator product expansion (2.3) is equivalent to the usual relations of the $N = 1$ supersymmetric extension of the Virasoro algebra,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}\left(m^3 - m\right)\delta_{m+n,0}\,, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}\,, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,,
\end{aligned} \tag{2.7}$$

where $m, n \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + 1/2$ for the NS algebra while $r, s \in \mathbb{Z}$ for the R algebra. In particular, the five operators L_0, $L_{\pm 1}$ and $G_{\pm 1/2}$ generate the orthosymplectic Lie algebra of the global superconformal group $\mathrm{OSp}(2,1)$.

In the simplest instance, logarithmic superconformal operators of weight Δ_C correspond to a pair of superfields

$$\begin{aligned}
\mathsf{C}(\mathsf{z}) &= C(z) + \theta\,\chi_C(z)\,, \\
\mathsf{D}(\mathsf{z}) &= D(z) + \theta\,\chi_D(z)
\end{aligned} \tag{2.8}$$

which have operator product expansions with the super energy-momentum tensor given by [15, 16]

$$\begin{aligned}
\mathsf{T}(\mathsf{z}_1)\,\mathsf{C}(\mathsf{z}_2) &= \frac{\Delta_C\,\theta_{12}}{(\mathsf{z}_{12})^2}\,\mathsf{C}(\mathsf{z}_2) + \frac{1}{2}\frac{1}{\mathsf{z}_{12}}\,\mathcal{D}_{\mathsf{z}_2}\mathsf{C}(\mathsf{z}_2) + \frac{\theta_{12}}{\mathsf{z}_{12}}\,\partial_{z_2}\mathsf{C}(\mathsf{z}_2) + \cdots\,, \\
\mathsf{T}(\mathsf{z}_1)\,\mathsf{D}(\mathsf{z}_2) &= \frac{\Delta_C\,\theta_{12}}{(\mathsf{z}_{12})^2}\,\mathsf{D}(\mathsf{z}_2) + \frac{\theta_{12}}{(\mathsf{z}_{12})^2}\,\mathsf{C}(\mathsf{z}_2) + \frac{1}{2}\frac{1}{\mathsf{z}_{12}}\,\mathcal{D}_{\mathsf{z}_2}\mathsf{D}(\mathsf{z}_2) + \frac{\theta_{12}}{\mathsf{z}_{12}}\,\partial_{z_2}\mathsf{D}(\mathsf{z}_2) + \cdots\,.
\end{aligned} \tag{2.9}$$

Note that $\mathsf{C}(\mathsf{z})$ is a primary superfield of the superconformal algebra of dimension Δ_C, which is necessarily an integer [3]. The appropriately normalized superfield $\mathsf{D}(\mathsf{z})$ is its quasi-primary logarithmic partner. This latter assumption, i.e. that $[L_n, \mathsf{D}(z)] = [G_r, \mathsf{D}(z)] = 0$ for $n, r > 0$, is not necessary, but it will simplify some of the arguments which follow. The operators $C(z)$ and $D(z)$ correspond to an ordinary logarithmic pair and their superpartners $\chi_C(z)$ and $\chi_D(z)$ are generated through the operator products with the fermionic supercurrent as

$$\begin{aligned}
G(z)\,C(z) &= \frac{1/2}{z-w}\,\chi_C(w) + \cdots\,, \\
G(z)\,D(z) &= \frac{1/2}{z-w}\,\chi_D(w) + \cdots\,.
\end{aligned} \tag{2.10}$$

In particular, in the NS algebra we may write the superpartners as $\chi_C(z) = [G_{-1/2}, C(z)]$ and $\chi_D(z) = [G_{-1/2}, D(z)]$.

2.2 Highest-weight representations

The quantum Hilbert space $\mathcal{H}$ of the superconformal field theory decomposes into two subspaces,

$$\mathcal{H} = \mathcal{H}_{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{R}}\,, \tag{2.11}$$

corresponding to the two types of boundary conditions obeyed by the fermionic fields. They carry the representations of the NS and R algebras, respectively. In this space, we assume that some of the highest-weight representations of the $N = 1$ superconformal algebra are indecomposable [11, 20]. Then a (rank 2) highest-weight Jordan cell of energy Δ_C is generated by a pair of appropriately normalized states $|C\rangle$, $|D\rangle$ obeying the conditions

$$\begin{aligned} L_0|C\rangle &= \Delta_C|C\rangle \,, \\ L_0|D\rangle &= \Delta_C|D\rangle + |C\rangle \,, \\ L_n|C\rangle &= L_n|D\rangle = 0 \,, \qquad n > 0 \,, \\ G_r|C\rangle &= G_r|D\rangle = 0 \,, \qquad r > 0 \,. \end{aligned} \tag{2.12}$$

A highest-weight representation of the logarithmic superconformal algebra is then generated by applying the raising operators L_n, G_r, $n, r < 0$ to these vectors giving rise to the descendant states of the theory. Note that $|C\rangle$ is a highest-weight state of the irreducible sub-representation of the superconformal algebra contained in the Jordan cell.

Neveu-Schwarz sector. The NS sector $\mathcal{H}_{\mathrm{NS}}$ of the Hilbert space contains the normalized, $\mathrm{OSp}(2,1)$-invariant vacuum state $|0\rangle$ which is the unique state of lowest energy $\Delta = 0$ in a unitary theory,

$$L_0|0\rangle = 0 \,. \tag{2.13}$$

In this sector, the states defined by (2.12) are in a one-to-one correspondence with the logarithmic operators satisfying the operator product expansions (2.9). Namely, under the usual operator-state correspondence of local quantum field theory, the superfields $\mathsf{C}(\mathsf{z})$ and $\mathsf{D}(\mathsf{z})$ are associated with highest weight states of energy Δ_C through

$$\begin{aligned} C(0)|0\rangle &= |C\rangle_{\mathrm{NS}} \,, \\ \chi_C(0)|0\rangle &= G_{-1/2}|C\rangle_{\mathrm{NS}} \,, \\ D(0)|0\rangle &= |D\rangle_{\mathrm{NS}} \,, \\ \chi_D(0)|0\rangle &= G_{-1/2}|D\rangle_{\mathrm{NS}} \,. \end{aligned} \tag{2.14}$$

In this way, the NS sector is formally analogous to an ordinary, bosonic logarithmic conformal field theory. Note that the vacuum state $|0\rangle$ itself corresponds to the identity operator I.

Ramond sector. Things are quite different in the R sector $\mathcal{H}_{\mathrm{R}}$. Consider a highest weight state $|\Delta\rangle_{\mathrm{R}}$ of energy Δ,

$$L_0|\Delta\rangle_{\mathrm{R}} = \Delta|\Delta\rangle_{\mathrm{R}} \,. \tag{2.15}$$

From the superconformal algebra (2.7), we see that the operators L_0 and G_0 commute in the R sector, so that the supercurrent zero mode G_0 acts on the highest weight states. As a consequence, the state $G_0|\Delta\rangle_{\mathrm{R}}$ also has energy Δ. Therefore, the highest weight states of the R sector $\mathcal{H}_{\mathrm{R}}$ come in orthogonal pairs $|\Delta\rangle_{\mathrm{R}}$, $G_0|\Delta\rangle_{\mathrm{R}}$ of the same energy. Under the operator-state correspondence, the Ramond highest weight states are created from the vacuum $|0\rangle$ by the application of spin fields $\Sigma^{\pm}_{\Delta}(z)$ [17] which are ordinary conformal fields of dimension Δ,

$$\begin{aligned}\Sigma^{+}_{\Delta}(0)|0\rangle &= |\Delta\rangle_{\rm R}\,, \\ \Sigma^{-}_{\Delta}(0)|0\rangle &= G_0|\Delta\rangle_{\rm R}\,.\end{aligned} \tag{2.16}$$

The operator product expansions of the spin fields with the super energy-momentum tensor may be computed from (2.15) and (2.16) and are given by

$$T(z)\,\Sigma^{\pm}_{\Delta}(w) = \frac{\Delta}{(z-w)^2}\,\Sigma^{\pm}_{\Delta}(w) + \frac{1}{z-w}\,\partial_w\Sigma^{\pm}_{\Delta}(w) + \cdots\,, \tag{2.17}$$

$$G(z)\,\Sigma^{+}_{\Delta}(w) = \frac{1}{2}\,\frac{1}{(z-w)^{3/2}}\,\Sigma^{-}_{\Delta}(w) + \cdots\,, \tag{2.18}$$

$$G(z)\,\Sigma^{-}_{\Delta}(w) = \frac{1}{2}\left(\Delta - \frac{\hat{c}}{16}\right)\frac{1}{(z-w)^{3/2}}\,\Sigma^{+}_{\Delta}(w) + \cdots\,, \tag{2.19}$$

where we have used the super-Virasoro algebra (2.7) to write

$$G_0^2 = L_0 - \frac{\hat{c}}{16}\,. \tag{2.20}$$

The operator product (2.17) merely states that $\Sigma^{\pm}_{\Delta}(z)$ is a dimension Δ primary field of the ordinary, bosonic Virasoro algebra, while (2.18) and (2.19) show that the fermionic supercurrent $G(z)$ is double-valued with respect to the spin fields, since they are equivalent to the monodromy conditions

$$G(\,{\rm e}^{\,2\pi\,{\rm i}}\,z)\,\Sigma^{\pm}_{\Delta}(w) = -G(z)\,\Sigma^{\pm}_{\Delta}(w)\,. \tag{2.21}$$

It follows that Ramond boundary conditions can be regarded as due to a branch cut in the complex plane connecting the spin fields $\Sigma^{\pm}_{\Delta}(z)$ at $z=0$ and $z=\infty$. The spin fields make the entire superconformal field theory non-local, and correspond to the irreducible representations of the Ramond algebra. Note that the ordinary superfields are block diagonal with respect to the decomposition (2.11), i.e. they are operators on $\mathcal{H}_{\rm NS}\to\mathcal{H}_{\rm NS}$ and $\mathcal{H}_{\rm R}\to\mathcal{H}_{\rm R}$, while the spin fields $\Sigma^{\pm}_{\Delta}:\mathcal{H}_{\rm NS}\to\mathcal{H}_{\rm R}$ are block off-diagonal.

The spin fields $\Sigma^{\pm}_{\Delta}(z)$ do not affect the integer weight fields $C(z)$ and $D(z)$, while their operator product expansions with the fermionic partners to the logarithmic operators in the R sector are given by

$$\begin{aligned}\chi_C(z)\,\Sigma^{\pm}_{\Delta}(w) &= \frac{1}{\sqrt{z-w}}\,\widetilde{\Sigma}^{\pm}_{C,\Delta}(w) + \cdots\,, \\ \chi_D(z)\,\Sigma^{\pm}_{\Delta}(w) &= \frac{1}{\sqrt{z-w}}\,\widetilde{\Sigma}^{\pm}_{D,\Delta}(w) + \cdots\,.\end{aligned} \tag{2.22}$$

The relations (2.22) define two different excited twist fields $\widetilde{\Sigma}^{\pm}_{C,\Delta}(z)$ and $\widetilde{\Sigma}^{\pm}_{D,\Delta}(z)$ which are conjugate to the spin fields $\Sigma^{\pm}_{\Delta}(z)$. They are also double-valued with respect to χ_C and χ_D, respectively, and they each act within the Ramond sector as operators on $\mathcal{H}_{\rm NS}\to\mathcal{H}_{\rm R}$. The relative non-locality of the operator product expansions (2.22) yields the global $\mathbb{Z}_2$-twists in the boundary conditions required of the R sector fermionic fields.

While $\widetilde{\Sigma}^{\pm}_{C,\Delta}(z)$ are primary fields of conformal dimension $\Delta_C+\Delta$, the conjugate spin fields $\widetilde{\Sigma}^{\pm}_{D,\Delta}(z)$ exhibit logarithmic mixing behaviour. This can be seen explicitly by applying the operator product expansions to both sides of (2.22) using (2.9) and (2.17)–(2.19) to get

$$T(z)\,\widetilde{\Sigma}^{\pm}_{C,\Delta}(w) = \frac{\Delta_C+\Delta}{(z-w)^2}\,\widetilde{\Sigma}^{\pm}_{C,\Delta}(w)+\frac{1}{z-w}\,\partial_w\widetilde{\Sigma}^{\pm}_{C,\Delta}(w)+\cdots\,, \tag{2.23}$$

$$T(z)\,\widetilde{\Sigma}^{\pm}_{D,\Delta}(w) = \frac{\Delta_C+\Delta}{(z-w)^2}\,\widetilde{\Sigma}^{\pm}_{D,\Delta}(w)+\frac{1}{(z-w)^2}\,\widetilde{\Sigma}^{\pm}_{C,\Delta}(w)+\frac{1}{z-w}\,\partial_w\widetilde{\Sigma}^{\pm}_{D,\Delta}(w)+\cdots\,, \tag{2.24}$$

$$G(z)\,\widetilde{\Sigma}^{+}_{C,\Delta}(w) = \frac{1}{2}\,\frac{1}{(z-w)^{3/2}}\,\widetilde{\Sigma}^{-}_{C,\Delta}(w)+\cdots\,, \tag{2.25}$$

$$G(z)\,\widetilde{\Sigma}^{-}_{C,\Delta}(w) = \frac{1}{2}\left(\Delta-\frac{\hat{c}}{16}\right)\frac{1}{(z-w)^{3/2}}\,\widetilde{\Sigma}^{+}_{C,\Delta}(w)+\cdots\,, \tag{2.26}$$

$$G(z)\,\widetilde{\Sigma}^{+}_{D,\Delta}(w) = \frac{1}{2}\,\frac{1}{(z-w)^{3/2}}\,\widetilde{\Sigma}^{-}_{D,\Delta}(w)+\cdots\,, \tag{2.27}$$

$$G(z)\,\widetilde{\Sigma}^{-}_{D,\Delta}(w) = \frac{1}{2}\left(\Delta-\frac{\hat{c}}{16}\right)\frac{1}{(z-w)^{3/2}}\,\widetilde{\Sigma}^{+}_{D,\Delta}(w)+\cdots\,. \tag{2.28}$$

The operator product expansions (2.23) and (2.24) yield a pair of ordinary, bosonic logarithmic conformal algebras, while (2.25)–(2.28) show that both $\widetilde{\Sigma}^{\pm}_{C,\Delta}(z)$ and $\widetilde{\Sigma}^{\pm}_{D,\Delta}(z)$ twist the fermionic supercurrent $G(z)$ in exactly the same way that the original spin fields $\Sigma^{\pm}_{\Delta}(z)$ do. In particular, the set of degenerate spin fields $\widetilde{\Sigma}^{\pm}_{C,\Delta}(z)$, $\widetilde{\Sigma}^{\pm}_{D,\Delta}(z)$ generate a pair of reducible but indecomposable representations (2.12) of the R algebra, of the *same* shifted weight $\Delta_C+\Delta$. The corresponding excited highest-weight states $|C,\Delta\rangle^{\pm}_{\rm R}$, $|D,\Delta\rangle^{\pm}_{\rm R}$ of the mutually orthogonal degenerate Jordan blocks for the action of the Virasoro operator L_0 on $\mathcal{H}_{\rm R}$ are created from the NS ground state through the application of the logarithmic spin operators as

$$\begin{aligned}\widetilde{\Sigma}^{\pm}_{C,\Delta}(0)|0\rangle &= |C,\Delta\rangle^{\pm}_{\rm R}\,,\\ \widetilde{\Sigma}^{\pm}_{D,\Delta}(0)|0\rangle &= |D,\Delta\rangle^{\pm}_{\rm R}\,,\end{aligned} \tag{2.29}$$

with

$$\begin{aligned}L_0|C,\Delta\rangle^{\pm}_{\rm R} &= (\Delta_C+\Delta)|C,\Delta\rangle^{\pm}_{\rm R}\,,\\ L_0|D,\Delta\rangle^{\pm}_{\rm R} &= (\Delta_C+\Delta)|D,\Delta\rangle^{\pm}_{\rm R}+|C,\Delta\rangle^{\pm}_{\rm R}\,.\end{aligned} \tag{2.30}$$

In the following we will be primarily interested in the spin fields associated with the Ramond ground state $|\Delta\rangle_{\rm R}$ which is defined by the condition $G_0|\Delta\rangle_{\rm R}=0$. This lifts the degeneracy of the highest weight representation which by (2.20) necessarily has dimension $\Delta=\hat{c}/16$, corresponding to the lowest energy in a unitary theory whereby $G_0^2\geq 0$. In this case, the Ramond state $G_0|\Delta\rangle_{\rm R}$ is a null vector and the R sector contains a single copy of the logarithmic superconformal algebra, as in the NS sector. We will return to the issue of logarithmic null vectors within this context in section 4. The spin field $\Sigma^{-}_{\hat{c}/16}(z)$ is then an irrelevant operator and may be set to zero, while the other spin field will be simply denoted by $\Sigma(z)\equiv\Sigma^{+}_{\hat{c}/16}(z)$. The spin field $\Sigma(z)$ corresponds to the unique supersymmetric ground state $|\frac{\hat{c}}{16}\rangle_{\mathrm{R}}$ of the Ramond system, with supersymmetry generator G_0, in the logarithmic superconformal field theory. Similarly, we may set $\widetilde{\Sigma}^-_{C,\hat{c}/16}(z) = \widetilde{\Sigma}^-_{D,\hat{c}/16}(z) = 0$, and we denote the remaining excited spin fields simply by $\widetilde{\Sigma}_C(z) \equiv \widetilde{\Sigma}^+_{C,\hat{c}/16}(z)$ and $\widetilde{\Sigma}_D(z) \equiv \widetilde{\Sigma}^+_{D,\hat{c}/16}(z)$.

3. Correlation functions

Carrying on with an abstract logarithmic superconformal algebra, we shall now describe the structure of logarithmic correlation functions in both the NS and R sectors. In particular, we will determine all two-point correlators involving the various logarithmic operators.

3.1 Ward identities and Neveu-Schwarz correlation functions

In the NS sector, we define the correlator of any periodic operator O as its vacuum expectation value

$$\langle \mathsf{O} \rangle_{\mathrm{NS}} = \langle 0|\mathsf{O}|0\rangle \,. \tag{3.1}$$

Such correlators of logarithmic operators, and their descendants, may be derived as follows. Consider a collection of Jordan blocks in the superconformal field theory of rank 2, weight Δ_{C_i}, and spanning logarithmic superfields $\mathsf{C}_i(\mathsf{z})$, $\mathsf{D}_i(\mathsf{z})$. Then, in the standard way, we may deduce from the operator product expansions (2.9) the superconformal Ward identities

$$\begin{aligned}
&\Big\langle \mathsf{T}(\mathsf{z})\, \mathsf{C}_n(\mathsf{z}_n)\cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\, \mathsf{D}_m(\mathsf{w}_m)\cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l})\Big\rangle_{\mathrm{NS}} = \\
&\quad = \left(\sum_{i=n}^{n+k}\left[\frac{1}{2}\,\frac{1}{z-z_i-\theta\,\theta_i}\,\mathcal{D}_{\mathsf{z}_i} + \frac{\theta-\theta_i}{z-z_i-\theta\,\theta_i}\,\partial_{z_i} + \frac{\Delta_{C_i}\,(\theta-\theta_i)}{(z-z_i-\theta\,\theta_i)^2}\right] + \right.\\
&\qquad \left. + \sum_{i=m}^{m+l}\left[\frac{1}{2}\,\frac{1}{z-w_i-\theta\,\zeta_i}\,\mathcal{D}_{\mathsf{w}_i} + \frac{\theta-\zeta_i}{z-w_i-\theta\,\zeta_i}\,\partial_{w_i} + \frac{\Delta_{C_i}\,(\theta-\zeta_i)}{(z-w_i-\theta\,\zeta_i)^2}\right]\right) \times \\
&\qquad \times \Big\langle \mathsf{C}_n(\mathsf{z}_n)\cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\, \mathsf{D}_m(\mathsf{w}_m)\cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l})\Big\rangle_{\mathrm{NS}} + \\
&\qquad + \sum_{i=m}^{m+l}\frac{\theta-\zeta_i}{(z-w_i-\theta\,\zeta_i)^2}\, \times \\
&\qquad \times \Big\langle \mathsf{C}_n(\mathsf{z}_n)\cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\mathsf{D}_m(\mathsf{w}_m)\cdots \mathsf{D}_{i-1}(\mathsf{w}_{i-1})\, \mathsf{C}_i(\mathsf{w}_i)\, \mathsf{D}_{i+1}(\mathsf{w}_{i+1})\cdots \\
&\qquad\qquad \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l})\Big\rangle_{\mathrm{NS}}\,,
\end{aligned} \tag{3.2}$$

where the supercoordinates in (3.2) are $\mathsf{z} = (z,\theta)$, $\mathsf{z}_i = (z_i,\theta_i)$ and $\mathsf{w}_i = (w_i,\zeta_i)$. These identities can be used to derive correlation functions of descendants of the logarithmic operators in terms of those involving the original superfields C_i and D_i. Notice, in particular, that the Ward identity connects amplitudes of the descendants of D_i with amplitudes involving the primary superfields C_i.

By expanding the super energy-momentum tensor into modes using (2.6) we may equate the coefficients on both sides of (3.2) corresponding to the actions of the $\mathrm{OSp}(2,1)$

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generators L_0, $L_{\pm 1}$ and $G_{\pm 1/2}$. By using global superconformal invariance of the vacuum state $|0\rangle$, we then arrive at a set of superfield differential equations

$$
\begin{aligned}
0 &= \left(\sum_{i=n}^{n+k} \mathcal{D}_{\mathsf{z}_i} + \sum_{i=m}^{m+l} \mathcal{D}_{\mathsf{w}_i} \right) \Big\langle \mathsf{C}_n(\mathsf{z}_n) \cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\, \mathsf{D}_m(\mathsf{w}_m) \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l}) \Big\rangle_{\rm NS} , \\
0 &= \left(\sum_{i=n}^{n+k} \Big[z_i\, \mathcal{D}_{\mathsf{z}_i} + \theta_i\, \partial_{\theta_i} + 2\Delta_{C_i} \Big] + \sum_{i=m}^{m+l} \Big[w_i\, \mathcal{D}_{\mathsf{w}_i} + \zeta_i\, \partial_{\zeta_i} + 2\Delta_{C_i} \Big] \right) \times \\
&\quad \times \Big\langle \mathsf{C}_n(\mathsf{z}_n) \cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\, \mathsf{D}_m(\mathsf{w}_m) \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l}) \Big\rangle_{\rm NS} + \\
&\quad + 2 \sum_{i=m}^{m+l} \Big\langle \mathsf{C}_n(\mathsf{z}_n) \cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k}) \times \\
&\qquad\qquad \times \mathsf{D}_m(\mathsf{w}_m) \cdots \mathsf{D}_{i-1}(\mathsf{w}_{i-1})\, \mathsf{C}_i(\mathsf{w}_i)\, \mathsf{D}_{i+1}(\mathsf{w}_{i+1}) \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l}) \Big\rangle_{\rm NS} , \\
0 &= \left(\sum_{i=n}^{n+k} \left[z_i^2\, \mathcal{D}_{\mathsf{z}_i} + z_i\, (\theta_i\, \partial_{\theta_i} + 2\Delta_{C_i}) \right] + \sum_{i=m}^{m+l} \left[w_i^2\, \mathcal{D}_{\mathsf{w}_i} + w_i\, (\zeta_i\, \partial_{\zeta_i} + 2\Delta_{C_i}) \right] \right) \times \\
&\quad \times \Big\langle \mathsf{C}_n(\mathsf{z}_n) \cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k})\, \mathsf{D}_m(\mathsf{w}_m) \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l}) \Big\rangle_{\rm NS} + \\
&\quad + 2 \sum_{i=m}^{m+l} w_i \Big\langle \mathsf{C}_n(\mathsf{z}_n) \cdots \mathsf{C}_{n+k}(\mathsf{z}_{n+k}) \times \\
&\qquad\qquad \times \mathsf{D}_m(\mathsf{w}_m) \cdots \mathsf{D}_{i-1}(\mathsf{w}_{i-1})\, \mathsf{C}_i(\mathsf{w}_i)\, \mathsf{D}_{i+1}(\mathsf{w}_{i+1}) \cdots \mathsf{D}_{m+l}(\mathsf{w}_{m+l}) \Big\rangle_{\rm NS} .
\end{aligned} \tag{3.3}
$$

These equations can be used to determine the general structure of the logarithmic correlators.

For the two-point correlation functions of the logarithmic superfields one finds [15]

$$\Big\langle \mathsf{C}(\mathsf{z}_1)\, \mathsf{C}(\mathsf{z}_2) \Big\rangle_{\rm NS} = 0 , \tag{3.4}$$

$$\Big\langle \mathsf{C}(\mathsf{z}_1)\, \mathsf{D}(\mathsf{z}_2) \Big\rangle_{\rm NS} = \Big\langle \mathsf{D}(\mathsf{z}_1)\, \mathsf{C}(\mathsf{z}_2) \Big\rangle_{\rm NS} = \frac{b}{(\mathsf{z}_{12})^{2\Delta_C}} , \tag{3.5}$$

$$\Big\langle \mathsf{D}(\mathsf{z}_1)\, \mathsf{D}(\mathsf{z}_2) \Big\rangle_{\rm NS} = \frac{1}{(\mathsf{z}_{12})^{2\Delta_C}} \Big(-2b \ln \mathsf{z}_{12} + d \Big) , \tag{3.6}$$

where the constant b is fixed by the leading logarithmic divergence of the conformal blocks of the theory (equivalently by the normalization of the D operator), and the integration constant d can be changed by the field redefinitions $\mathsf{D}(\mathsf{z}) \mapsto \mathsf{D}(\mathsf{z}) + \lambda\, \mathsf{C}(\mathsf{z})$ which are induced by the scale transformations $z \mapsto {\rm e}^{\,\lambda}\, z$. In particular, the equality of two-point functions in (3.5) immediately implies that the conformal dimension Δ_C of the logarithmic pair is necessarily an integer [3]. For the three-point functions one gets [15]

$$\Big\langle \mathsf{C}(\mathsf{z}_1)\, \mathsf{C}(\mathsf{z}_2)\, \mathsf{C}(\mathsf{z}_3) \Big\rangle_{\rm NS} = 0 , \tag{3.7}$$

$$\Big\langle \mathsf{C}(\mathsf{z}_1)\, \mathsf{C}(\mathsf{z}_2)\, \mathsf{D}(\mathsf{z}_3) \Big\rangle_{\rm NS} = \frac{1}{(\mathsf{z}_{12})^{\Delta_C}\, (\mathsf{z}_{13})^{\Delta_C}\, (\mathsf{z}_{23})^{\Delta_C}} \Big(b_1 + \beta_1\, \theta_{123} \Big) , \tag{3.8}$$

$$\Big\langle \mathsf{C}(\mathsf{z}_1)\, \mathsf{D}(\mathsf{z}_2)\, \mathsf{D}(\mathsf{z}_3) \Big\rangle_{\rm NS} = \frac{1}{(\mathsf{z}_{12})^{\Delta_C} (\mathsf{z}_{13})^{\Delta_C} (\mathsf{z}_{23})^{\Delta_C}} \Big(b_2 + \beta_2 \theta_{123} - 2(b_1 + \beta_1 \theta_{123}) \ln \mathsf{z}_{23} \Big) , \tag{3.9}$$

$$
\Big\langle \mathsf{D}(\mathsf{z}_1)\,\mathsf{D}(\mathsf{z}_2)\,\mathsf{D}(\mathsf{z}_3)\Big\rangle_{\rm NS} = \frac{1}{(\mathsf{z}_{12})^{\Delta_C}\,(\mathsf{z}_{13})^{\Delta_C}\,(\mathsf{z}_{23})^{\Delta_C}} \times
\times\Big[b_3 + \beta_3\,\theta_{123} - \;(b_2 + \beta_2\,\theta_{123})\ln \mathsf{z}_{12}\,\mathsf{z}_{13}\,\mathsf{z}_{23} + (b_1 + \beta_1\,\theta_{123}) \times
\times\,(2\ln \mathsf{z}_{12}\ln \mathsf{z}_{13} + 2\ln \mathsf{z}_{12}\ln \mathsf{z}_{23} + 2\ln \mathsf{z}_{13}\ln \mathsf{z}_{23} - \ln^2 \mathsf{z}_{12} -
- \ln^2 \mathsf{z}_{13} - \ln^2 \mathsf{z}_{23})\Big]\,, \tag{3.10}
$$

where b_i and β_i are undetermined Grassmann even and odd constants, respectively, and we have generally defined

$$
\theta_{ijk} = \frac{1}{\sqrt{\mathsf{z}_{ij}\,\mathsf{z}_{jk}\,\mathsf{z}_{ki}}}\Big(\theta_i\,\mathsf{z}_{jk} + \theta_j\,\mathsf{z}_{ki} + \theta_k\,\mathsf{z}_{ij} + \theta_i\,\theta_j\,\theta_k\Big)\,. \tag{3.11}
$$

The remaining three-point correlation functions can be obtained via cyclic permutation of the superfields in (3.8) and (3.9). The general form of the four-point functions may also be found in [15].

3.2 Ramond correlation functions

In the R sector, we define the correlator of any operator O to be its normalized expectation value in the supersymmetric Ramond ground state,

$$
\langle \mathsf{O}\rangle_{\rm R} = \frac{\langle 0|\Sigma(\infty)\,\mathsf{O}\,\Sigma(0)|0\rangle}{\langle 0|\Sigma(\infty)\,\Sigma(0)|0\rangle}\,, \tag{3.12}
$$

where we have used the standard asymptotic out-state definition

$$
\langle 0|\Sigma(\infty) = \lim_{z\to\infty}\langle 0|\Sigma(z)\,z^{\hat{c}/8} \tag{3.13}
$$

and the fact that the spin field $\Sigma(z)$ is a primary field of the ordinary Virasoro algebra of dimension $\Delta = \hat{c}/16$. In particular, the two-point function of the (appropriately normalized) spin operator is given by

$$
\langle 0|\Sigma(z)\,\Sigma(w)|0\rangle = \frac{1}{(z-w)^{\hat{c}/8}}\,. \tag{3.14}
$$

Since $\Sigma(z)$ does not act on the bosonic fields $C(z)$ and $D(z)$, their R sector correlation functions coincide with those of the NS sector, i.e. with those of an ordinary logarithmic conformal field theory. In particular, for the two-point functions we find [1, 3]

$$
\begin{aligned}
\Big\langle C(z)\,C(w)\Big\rangle_{\rm R} &= 0\,,\\
\Big\langle C(z)\,D(w)\Big\rangle_{\rm R} &= \Big\langle D(z)\,C(w)\Big\rangle_{\rm R} \;=\; \frac{b}{(z-w)^{2\Delta_C}}\,,\\
\Big\langle D(z)\,D(w)\Big\rangle_{\rm R} &= \frac{d - 2b\ln(z-w)}{(z-w)^{2\Delta_C}}\,.
\end{aligned} \tag{3.15}
$$

For the correlation functions of the fermionic fields, we proceed as follows. Let us introduce the function

$$
g_C(z,w|z_1,z_2) = \frac{\langle 0|\Sigma(z_1)\,\chi_C(z)\,\chi_C(w)\,\Sigma(z_2)|0\rangle}{\langle 0|\Sigma(z_1)\,\Sigma(z_2)|0\rangle}\,. \tag{3.16}
$$

All fields appearing in (3.16) behave as ordinary primary fields under the action of the Virasoro algebra. The Green's function (3.16) can therefore be evaluated using standard conformal field theoretic methods [26]. It obeys the asymptotic conditions

$$g_C(z,w|z_1,z_2) \simeq 0 + \cdots \qquad \text{as} \quad z \to w \, , \tag{3.17}$$
$$\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{z-z_1}} \langle 0|\widetilde{\Sigma}_C(z_1)\, \chi_C(w)\, \Sigma(z_2)|0\rangle + \cdots \qquad \text{as} \quad z \to z_1 \, , \tag{3.18}$$
$$\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{z-z_2}} \langle 0|\Sigma(z_1)\, \chi_C(w)\, \widetilde{\Sigma}_C(z_2)|0\rangle + \cdots \qquad \text{as} \quad z \to z_2 \, . \tag{3.19}$$

The first condition (3.17) arises from the fact that the short distance behaviour of the quantum field theory is independent of the global boundary conditions, so that in the limit $z \to w$ the function (3.16) should coincide with the corresponding Neveu-Schwarz two-point function determined in (3.4), i.e. $\langle \chi_C(z)\, \chi_C(w)\rangle_{\rm NS} = 0$. The local monodromy conditions (3.18) and (3.19) follow from the operator product expansions (2.22). In addition, by Fermi statistics the Green's function (3.16) must be antisymmetric under the exchange of its arguments z and w,

$$g_C(z,w|z_1,z_2) = -g_C(w,z|z_1,z_2) \, . \tag{3.20}$$

By translation invariance, the conditions (3.17) and (3.20) are solved by any odd analytic function f of $z-w$. Since the correlators appearing in (3.18) and (3.19) involve only ordinary, primary conformal fields, global conformal invariance dictates that the function $f(z-w)$ must multiply a quantity which is a function only of the $SL(2,\mathbb{C})$-invariant anharmonic ratio x of the four points of $g_C(z,w|z_1,z_2)$ given by

$$x = \frac{(z-z_1)(w-z_2)}{(z-z_2)(w-z_1)} \, . \tag{3.21}$$

By conformal invariance, the odd analytic function $f(z-w)$ is therefore identically 0, and hence

$$g_C(z,w|z_1,z_2) = 0 \, . \tag{3.22}$$

Using this result we can determine a number of correlation functions. Setting $z_1 = \infty$ and $z_2 = 0$ gives the Ramond correlator

$$\Big\langle \chi_C(z)\, \chi_C(w) \Big\rangle_{\rm R} = 0 \, . \tag{3.23}$$

From (3.19) and (3.22) we obtain in addition the vanishing mixed correlator

$$\langle 0|\Sigma(z_1)\, \chi_C(z_2)\, \widetilde{\Sigma}_C(z_3)|0\rangle = 0 \, . \tag{3.24}$$

Fusing together the fields $\Sigma(z_1)$ and $\chi_C(z_2)$ in (3.24) using (2.22) then gives the conjugate spin-spin correlator

$$\langle 0|\widetilde{\Sigma}_C(z)\, \widetilde{\Sigma}_C(w)|0\rangle = 0 \, . \tag{3.25}$$

The vanishing of the $\widetilde{\Sigma}_C\widetilde{\Sigma}_C$ correlation function is consistent with the fact that the excited spin field $\widetilde{\Sigma}_C(z)$ obeys the logarithmic conformal algebra (2.23), (2.24) [1, 3].

Next, let us consider the function

$$g_D(z,w|z_1,z_2) = \frac{\langle 0|\Sigma(z_1)\,\chi_C(z)\,\chi_D(w)\,\Sigma(z_2)|0\rangle}{\langle 0|\Sigma(z_1)\,\Sigma(z_2)|0\rangle}\,. \tag{3.26}$$

The action of the Virasoro algebra in (3.26) does not produce any additional terms from the logarithmic mixing of the fermionic field $\chi_D(w)$, because of the vanishing property (3.22). Therefore, this function can also be evaluated as if the theory were an ordinary conformal field theory [26]. Using (2.22) and (3.5) the asymptotic conditions (3.17)–(3.19) are now replaced with

$$\begin{aligned} g_D(z,w|z_1,z_2) &\simeq \frac{2\Delta_C\,b}{(z-w)^{2\Delta_C+1}} + \cdots \qquad \text{as} \quad z\to w\,, & (3.27)\\ &\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{z-z_1}}\,\langle 0|\widetilde{\Sigma}_C(z_1)\,\chi_D(w)\,\Sigma(z_2)|0\rangle + \cdots \qquad \text{as} \quad z\to z_1\,, & (3.28)\\ &\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{z-z_2}}\,\langle 0|\Sigma(z_1)\,\chi_D(w)\,\widetilde{\Sigma}_C(z_2)|0\rangle + \cdots \qquad \text{as} \quad z\to z_2\,, & (3.29)\\ &\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{w-z_1}}\,\langle 0|\widetilde{\Sigma}_D(z_1)\,\chi_C(z)\,\Sigma(z_2)|0\rangle + \cdots \qquad \text{as} \quad w\to z_1\,, & (3.30)\\ &\simeq \frac{(z_1-z_2)^{\hat{c}/8}}{\sqrt{w-z_2}}\,\langle 0|\Sigma(z_1)\,\chi_C(z)\,\widetilde{\Sigma}_D(z_2)|0\rangle + \cdots \qquad \text{as} \quad w\to z_2\,. & (3.31) \end{aligned}$$

Again, from (3.24) it follows that the correlators in (3.28)–(3.31) involving a single logarithmic operator can be treated as an ordinary conformal correlator for primary fields. In particular, we can treat (3.26) as a correlator for two identical conformal fermion fields of dimension $\Delta_C + \frac{1}{2}$ and require it to be antisymmetric under exchange of z and w, as in (3.20). This property follows from the fact that the local NS correlator (3.27) is antisymmetric in z and w and this feature should extend globally in the quantum field theory. Again, by SL$(2,\mathbb{C})$-invariance the quantity $(z-w)^{-2\Delta_C-1}\,g_D(z,w|z_1,z_2)$ is a function only of the anharmonic ratio (3.21). The precise dependence on x is uniquely determined by the boundary conditions (3.27)–(3.31) and the antisymmetry of g_D, and we find[1]

$$g_D(z,w|z_1,z_2) = \frac{\Delta_C\,b}{(z-w)^{2\Delta_C+1}}\left(\sqrt{\frac{(z-z_1)(w-z_2)}{(z-z_2)(w-z_1)}} + \sqrt{\frac{(z-z_2)(w-z_1)}{(z-z_1)(w-z_2)}}\right). \tag{3.32}$$

By taking various limits of (3.32) we can generate another set of correlation functions for Ramond sector operators. In the simultaneous limit $z_1\to\infty$ and $z_2\to 0$, the

[1] To show explicitly that (3.32) is the unique function of z and w with the desired properties, we write it as

$$g_D(z,w|z_1,z_2) = \frac{1}{\sqrt{(z-z_1)(z-z_2)(w-z_1)(w-z_2)}}\,\frac{\Delta_C\,b}{(z-w)^{2\Delta_C+1}}\Big((z-z_1)(w-z_2)+(z-z_2)(w-z_1)\Big)\,.$$

The first factor here gives the correct behaviour for g_D as $z,w\to z_1,z_2$, while the second factor is the required pole at $z=w$ of order $2\Delta_C+1$. The third factor is then chosen so that the residue of the pole is $2\Delta_C\,b$ and such that it cancels the lower order poles arising from the first factor in the limit $z\to w$, and by further requiring that the overall combination be antisymmetric in z and w.

function (3.32) yields the Ramond two-point correlators

$$\Big\langle \chi_C(z)\,\chi_D(w)\Big\rangle_{\rm R} = -\Big\langle \chi_D(z)\,\chi_C(w)\Big\rangle_{\rm R} = \frac{\Delta_C\, b}{(z-w)^{2\Delta_C+1}}\,\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\,\right)\,. \tag{3.33}$$

Note that the term in parantheses has branch cuts at $z=0,\infty$ and $w=0,\infty$, yielding the antiperiodic boundary conditions on the spinor fields as they circle around the origin in the complex plane and across the cut connecting the spin operators $\Sigma(0)$ and $\Sigma(\infty)$ in (3.12). Taking the limits $z\to z_1,z_2$ and $w\to z_1,z_2$ in (3.32) and comparing with (3.28)–(3.31) yields the correlation functions

$$\begin{aligned}\langle 0|\widetilde{\Sigma}_C(z_1)\,\chi_D(z_2)\,\Sigma(z_3)|0\rangle &= \frac{{\rm i}\,\Delta_C\, b}{(z_1-z_2)^{2\Delta_C+1/2}\,(z_1-z_3)^{\hat c/8-1/2}\,\sqrt{z_2-z_3}}\\ &= -\,\langle 0|\widetilde{\Sigma}_D(z_1)\,\chi_C(z_2)\,\Sigma(z_3)|0\rangle\,.\end{aligned} \tag{3.34}$$

Fusing $\Sigma(z_3)$ with $\chi_D(z_2)$ and $\chi_C(z_2)$ in (3.34) using (2.22) then yields the spin-spin correlators

$$\langle 0|\widetilde{\Sigma}_C(z)\,\widetilde{\Sigma}_D(w)|0\rangle = -\,\langle 0|\widetilde{\Sigma}_D(z)\,\widetilde{\Sigma}_C(w)|0\rangle = \frac{{\rm i}\,\Delta_C\, b}{(z-w)^{2\Delta_C+\hat c/8}}\,. \tag{3.35}$$

Note that the logarithmic pair $\widetilde{\Sigma}_C,\widetilde{\Sigma}_D$ does not have the canonical two-point functions of a logarithmic conformal field theory (see (3.15)). This is because the excited spin fields of the theory are not bosonic fields, but are rather given by non-local operators which interpolate between different sectors of the quantum Hilbert space and which satisfy, in addition to the logarithmic algebra, a supersymmetry algebra. In fact, their correlators are almost identical in form to the correlation functions of the logarithmic superpartners χ_C,χ_D [16].

Finally, we need to compute the DD type correlators. The above techniques do not directly apply because Green's functions with two or more logarithmic operator insertions will not transform covariantly under the action of the Virasoro algebra. However, we may obtain the DD type correlators from the mixed CD type ones above by the following trick [15, 16, 27]. We regard Δ_C as a continuous weight and note that the logarithmic superconformal algebra can be simply obtained by writing down the standard conformal operator product expansions for the C type operators, and then differentiating them with respect to Δ_C to obtain the D type ones with the formal identifications $D=\partial C/\partial\Delta_C$, $\chi_D=\partial\chi_C/\partial\Delta_C$ and $\widetilde{\Sigma}_D=\partial\widetilde{\Sigma}_C/\partial\Delta_C$. Since the basic spin fields $\Sigma(z)$ do not depend on the conformal dimension Δ_C, we can differentiate the correlation functions (3.33)–(3.35) to get the desired Green's functions. In doing so we regard the parameter b as an analytic function of the weight Δ_C and define $d=\partial b/\partial\Delta_C$. In this way we arrive at the correlators

$$\begin{aligned}\Big\langle \chi_D(z)\,\chi_D(w)\Big\rangle_{\rm R} &= \frac{b+\Delta_C\Big(d-2b\ln(z-w)\Big)}{(z-w)^{2\Delta_C+1}}\,\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\,\right)\,,\\ \langle 0|\widetilde{\Sigma}_D(z_1)\,\chi_D(z_2)\,\Sigma(z_3)|0\rangle &= -\frac{b+\Delta_C\Big(d-2b\ln(z_1-z_2)\Big)}{{\rm i}\,(z_1-z_2)^{2\Delta_C+1/2}\,(z_1-z_3)^{\hat c/8-1/2}\,\sqrt{z_2-z_3}}\,,\\ \langle 0|\widetilde{\Sigma}_D(z)\,\widetilde{\Sigma}_D(w)|0\rangle &= -\frac{b+\Delta_C\Big(d-2b\ln(z-w)\Big)}{{\rm i}\,(z-w)^{2\Delta_C+\hat c/8}}\,.\end{aligned} \tag{3.36}$$

In a completely analogous way, we may easily determine the vanishing two-point correlation functions

$$\Big\langle \phi(z)\,\chi_{\phi'}(w)\Big\rangle_{\rm R} ~=~ 0 ~=~ \langle 0|\widetilde{\Sigma}_{\phi'}(z_1)\,\phi(z_2)\,\Sigma(z_3)|0\rangle\,, \tag{3.37}$$

where ϕ and ϕ' label either of the two fields C or D. The present technique unfortunately does not directly determine higher order correlation functions of the fields. As they will not be required in what follows, we will not pursue this issue in this paper.

4. Null vectors, hidden symmetries and spin models

It has been suggested [16] that, in the limit $\Delta_C = 0$, the fermionic field $\chi_C(z)$ in (2.8) may be a null field, since its two-point correlation functions with all other logarithmic fields vanish for zero conformal dimension. Furthermore, the logarithmic scaling violations in the fermionic two-point functions involving the field $\chi_D(z)$ disappear in this limit. While this latter property is certainly true for all Green's functions of the conformal field theory, a quick examination of the three-point correlators (3.7) and (3.8) shows that $\chi_C(z)$ is *not* a null field if $\beta_1 \neq 0$. The situation is completely analogous to what happens generically to its superpartner $C(z)$. Since the primary field $C(z)$ creates a zero-norm state, and since $\Delta_C \in \mathbb{Z}$, there is a new hidden continuous symmetry in the theory [3] generated by the conserved holomorphic current $C(z)$, which is a symmetric tensor of rank Δ_C. For $\Delta_C = 0$, the extra couplings of the χ_C field for $\beta_1 \neq 0$ show that it corresponds to a non-trivial, dynamical fermionic symmetry of the logarithmic superconformal field theory. In fact, in the R sector the structure of these continuous symmetries is even richer, given that the excited spin field $\widetilde{\Sigma}_C(z)$ also creates a zero-norm state in the logarithmic superconformal field theory, and that it has vanishing two-point functions for $\Delta_C = 0$. In $c \neq 0$ theories where the bosonic energy-momentum tensor $T(z)$ has a logarithmic partner, the identity operator I generates a Jordan cell with $\Delta_C = 0$ [19] and the zero-norm state is the vacuum, $\langle 0|0\rangle = 0$. In this case, of course, the fermion field $\chi_I(z) = 0$ is trivially a null field, and its partner $\chi_D(z)$ is an ordinary, non-logarithmic primary field of the Virasoro algebra of conformal dimension 1/2. Similarly, in this case $\widetilde{\Sigma}_C(z) = 0$, while $\widetilde{\Sigma}_D(z)$ is an ordinary, non-logarithmic twist field of weight $\hat{c}/16$.

In the Ramond sector, there are natural ways to generate null states for any Δ_C. One way is to build the representation of the Ramond algebra from the supersymmetric ground state $|\hat{c}/16\rangle_{\rm R}$ as described at the end of section 2.2. Another way is to introduce the fermion parity operator $\Gamma = (-1)^F$, where F is the fermion number operator of the superconformal field theory. The operator Γ commutes with integer spin fields and anticommutes with half-integer spin fields. It defines an inner automorphism $\pi_\Gamma : \mathcal{C} \to \mathcal{C}$ of the maximally extended chiral symmetry algebra $\mathcal{C}$ of the superconformal field theory, such that there is an exact sequence of vector spaces

$$0 ~\longrightarrow~ \mathcal{C}^+ ~\longrightarrow~ \mathcal{C} ~\longrightarrow~ \mathcal{C}^- ~\longrightarrow~ 0\,, \qquad \pi_\Gamma(\mathcal{C}^\pm) = \pm\,\mathcal{C}^\pm\,. \tag{4.1}$$

Under the operator-state correspondence, this determines a fermion parity grading of the Hilbert space of states as

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \, , \qquad \Gamma \, \mathcal{H}^\pm = \pm \, \mathcal{H}^\pm \, . \tag{4.2}$$

Since G_0 reverses chirality, the paired Ramond ground states have opposite chirality,

$$\Gamma \, \Sigma_\Delta^\pm(0)|0\rangle = \pm \, \Sigma_\Delta^\pm(0)|0\rangle \, . \tag{4.3}$$

The opposite chirality spin fields $\Sigma_\Delta^\pm(z)$ are non-local with respect to each other (c.f. (2.18) and (2.19)). In a unitary theory, whereby $G_0^2 \geq 0$, all $\Delta = \hat{c}/16$ states are chirally asymmetric highest-weight states, since the state $G_0|\Delta\rangle_{\rm R}$ is then a null vector in the Hilbert space. On the other hand, the orthogonal projection $(1/2)\,(1+\Gamma) : \mathcal{H} \to \mathcal{H}^+$ onto states of even fermion parity $\Gamma = 1$ eliminates the spin field $\Sigma_\Delta^-(z)$ and gives a local field theory which is customarily refered to as a "spin model" [17]. The fields of the spin model live in the local chiral algebra $\mathcal{C}^+$. This projection eliminates $G(z)$ and the other half-integer weight fields. When combined with the projection onto $G_0 = 0$ it gives the "GSO projection" which will be important in the D-brane applications of the next section 5.

The main significance of the chiral subalgebra restriction $(1/2)\,(1 + \pi_\Gamma) : \mathcal{C} \to \mathcal{C}^+$ is that the fermionic fields of the superconformal field theory can be reconstructed from the $\Gamma = 1$ spin fields $\Sigma(z)$, at least in the examples that we consider in this paper. In an analogous way, the logarithmic superpartners $\chi_C(z)$ and $\chi_D(z)$ can be reconstructed from the $\Gamma = 1$ excited spin fields $\widetilde{\Sigma}_C(z)$ and $\widetilde{\Sigma}_D(z)$. By supersymmetry, this yields the bosonic partners $C(z)$ and $D(z)$, and so in this way the spin model determines the entire logarithmic superconformal field theory. In fact, the spin field $\Sigma(z)$ can be uniquely constructed from the underlying chiral current algebra generated by currents which are formed by the primary fermionic fields of the theory [28]. The fermionic current algebra will thereby completely determine the entire logarithmic superconformal field theory.

5. The Recoil problem in superstring theory

In the remainder of this paper we will consider some concrete models to illustrate the above formalism explicitly. These examples will also serve to describe some of the basic constructions of logarithmic spin operators and will illustrate the applicability of the superconformal logarithmic formalism. In this section we will discuss the logarithmic superconformal field theory that describes the recoil of a D-particle in string theory [7, 8, 16]. This is the simplest example in which to introduce some of the formalism that will also be used in later sections. It also captures the essential features of the general theory of the previous three sections in a very simple setting.

5.1 Supersymmetric impulse operators

Consider the superconformal field theory defined by the classical worldsheet action

$$S_{\rm D0} = \frac{1}{2\pi} \int {\rm d}z \ {\rm d}\overline{z} \ {\rm d}\theta \ {\rm d}\overline{\theta} \ \ \overline{\mathcal{D}_{\sf z}} \, {\sf x}^\mu \, \mathcal{D}_{\sf z} {\sf x}_\mu - \frac{1}{\pi} \oint {\rm d}\tau \ {\rm d}\vartheta \ \Big(y_i \, {\sf C}_\epsilon + u_i \, {\sf D}_\epsilon \Big) \, \mathcal{D}_\perp {\sf x}^i \, , \tag{5.1}$$

where $\mathsf{x}^\mu(\mathsf{z},\bar{\mathsf{z}}) = \mathsf{x}^\mu(\mathsf{z}) + \mathsf{x}^\mu(\bar{\mathsf{z}})$ with $\mathsf{x}^\mu(\mathsf{z})$ the chiral scalar superfield

$$\mathsf{x}^\mu(\mathsf{z}) = x^\mu(z) + \theta\,\psi^\mu(z)\,, \tag{5.2}$$

whose Neveu-Schwarz two-point functions are given by

$$\Big\langle \mathsf{x}^\mu(\mathsf{z}_1)\,\mathsf{x}^\nu(\mathsf{z}_2)\Big\rangle_{\rm NS} = -\delta^{\mu\nu}\,\ln \mathsf{z}_{12}\,. \tag{5.3}$$

Here x^μ, $\mu = 1,\dots,10$ are maps from the upper complex half-plane $\mathbb{C}_+$ into ten dimensional *euclidean* space $\mathbb{R}^{10}$, and ψ^μ are their spin 1/2 fermionic superpartners that transform in the vector representation of SO(10) and each of which is a Majorana-Weyl spinor in two-dimensions. We will identify the coordinate x^{10} as the euclidean time, while x^i, $i = 1,\dots,9$ lie along the spatial directions in the target space of the open strings. As in the previous section, we concentrate on the chiral sector of the worldsheet field theory with superfields (5.2). The chiral super energy-momentum tensor is given by

$$\mathsf{T}(\mathsf{z}) = -\frac{1}{2}\,\mathcal{D}_{\mathsf{z}}\mathsf{x}^\mu(\mathsf{z})\,\partial_z\mathsf{x}_\mu(\mathsf{z})\,. \tag{5.4}$$

The reasons for working with euclidean spacetime signature are technical. First of all, it is easier to deal with spinor representations of the euclidean group SO(10) than with those of the Lorentz group SO(9,1). In the former case all of the ψ^μ are treated on equal footing and one is free from the possible complications arising from the time-like nature of x^0, which would otherwise imply a special role for its superpartner ψ^0 [29]. Secondly, for the recoil problem, euclidean target spaces are necessary to ensure convergence of worldsheet correlation functions among the logarithmic operators [7]. For calculational definiteness and convenience of the worldsheet path integrals, we shall therefore adopt a euclidean signature convention in the following.

The second term in the action (5.1) is a marginal deformation of the free $\hat{c} = 10$ superconformal field theory by the vertex operator describing the recoil, within an impulse approximation, of a non-relativistic D-brane in target space due to its interaction with closed string scattering states [6]–[8]. It is the appropriate operator to use when regarding the branes as string solitons. The coordinate τ parametrizes the boundary of the upper half-plane, and ϑ is a real Grassmann coordinate. The fields in this part of the action are understood to be restricted to the worldsheet boundary. The coupling constants y_i and u_i are interpreted as the initial position and constant velocity of the D-particle, respectively, and the subscript $\perp$ denotes differentiation in the direction normal to the boundary of $\mathbb{C}_+$. The recoil operators are given by chiral superfields $\mathsf{C}_\epsilon(\mathsf{z})$ and $\mathsf{D}_\epsilon(\mathsf{z})$ whose components are defined in terms of superpositions over tachyon vertex operators $\mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)}$ in the time direction as [16]

$$C_\epsilon(z) = \frac{\epsilon}{4\pi\,\mathrm{i}}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{q-\mathrm{i}\,\epsilon}\;\mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)}\,,$$
$$\chi_{C_\epsilon}(z) = \mathrm{i}\,\epsilon\,C_\epsilon(z)\otimes\psi^{10}(z)\,,$$

$$D_\epsilon(z) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}q}{(q-\mathrm{i}\,\epsilon)^2}\ \mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)}\,,$$

$$\chi_{D_\epsilon}(z) = \mathrm{i}\,\left(\epsilon\,D_\epsilon(z) - \frac{2}{\epsilon}\,C_\epsilon(z)\right)\otimes\psi^{10}(z)\,. \tag{5.5}$$

Here and in the following, singular operator products taken at coincident points are always understood to be normal ordered according to the prescription

$$O(z)\,O'(z) \equiv \oint\limits_{w=z}\frac{\mathrm{d}w}{2\pi\,\mathrm{i}}\ \frac{O(w)\,O'(z)}{w-z}\,. \tag{5.6}$$

The target space regularization parameter $\epsilon \to 0^+$ is related to the worldsheet ultraviolet cutoff $\Lambda \to 0^+$ by

$$\frac{1}{\epsilon^2} = -\ln\Lambda\,. \tag{5.7}$$

In this limit, careful computations [7, 16] establish that, to leading orders in ϵ, the superfield recoil operators (5.5) satisfy the relations (2.9) and (3.4)–(3.6) of the $N = 1$ logarithmic superconformal algebra in the NS sector of the worldsheet field theory, with

$$\begin{aligned}\Delta_{C_\epsilon} &= -\frac{\epsilon^2}{4}\,,\\ b &= \frac{\pi^{3/2}}{4}\,,\\ d &= \frac{\pi^{3/2}}{2\epsilon^2}\,.\end{aligned} \tag{5.8}$$

In the following we will describe how to properly incorporate the Ramond sector of this system.

5.2 Spin fields

We will now construct the operators Σ which create cuts in the fields ψ^{10} appearing in the superpartners of the recoil operators (5.5) and are thereby responsible for changing their boundary conditions as one circumnavigates the cut [29]. In fact, one needs $\Sigma(z)$ in the neighborhood of the fields ψ^{10} but this is readily done in bosonized form [30], as we shall now discuss, by means of a boson translation operator which relates $\Sigma(z)$ to $\Sigma(0)$. Bosonization of the free fermion system defined by (5.1) allows us to express in a local-looking form the non-local effects of the spin operators. In what follows we shall only require the bosonization of the spinor field appearing in (5.5).

In the euclidean version of the target space theory there are ten fermion fields ψ^μ which we can treat on equal footing. Given the pair of right-moving NSR fermion fields ψ^9, ψ^{10} corresponding to the light-cone of the recoiling D0-brane system, we may form complex Dirac fermion fields

$$\psi^\pm(z) = \psi^9(z) \pm \mathrm{i}\,\psi^{10}(z)\,. \tag{5.9}$$

The worldsheet kinetic energy in (5.1) associated to this pair is of the form

$$\int \mathrm{d}^2z\ \left(\psi^9\,\overline{\partial_z}\,\psi^9 + \psi^{10}\,\overline{\partial_z}\,\psi^{10}\right) = \int \mathrm{d}^2z\ \psi^+\,\overline{\partial_z}\,\psi^-\,. \tag{5.10}$$

From the corresponding equations of motion and (5.3) it follows that the field

$$j(z) = \psi^+(z)\,\psi^-(z) \tag{5.11}$$

is a conserved U(1) fermion number current which is a primary field of the Virasoro algebra of dimension 1 and which generates a U(1) current algebra at level 1. Its presence allows the introduction of spin fields, and hence twisted sectors in the quantum Hilbert space, through the bosonization formulas

$$\begin{aligned} j(z) &= 2\,\mathrm{i}\;\partial_z\phi(z)\,, \\ \psi^\pm(z) &= \sqrt{2}\;\mathrm{e}^{\,\pm\mathrm{i}\,\phi(z)}\,, \end{aligned} \tag{5.12}$$

where $\phi(z)$ is a free, real, compact chiral scalar field, i.e. its two-point function is

$$\langle 0|\phi(z)\,\phi(w)|0\rangle = -\ln(z-w)\,. \tag{5.13}$$

In this representation all fields are taken to act in the NS sector.

The holomorphic part of the Sugarawa energy-momentum tensor corresponding to the worldsheet action (5.10) is given in bosonized form by

$$T_\kappa(z) = -\frac{1}{2}\,\partial_z\phi(z)\,\partial_z\phi(z) + \frac{\mathrm{i}\,\kappa}{2}\,\partial_z^2\phi(z)\,, \tag{5.14}$$

where the constant κ is arbitrary because the second term in (5.14) is identically conserved for all κ. This energy-momentum tensor derives from the Coulomb gas model defined by the Liouville action

$$S_\kappa = \frac{1}{4\pi}\int \mathrm{d}z\;\mathrm{d}\overline{z}\;\sqrt{g}\;\left(\partial_z\phi\,\overline{\partial_z}\,\phi + \frac{\mathrm{i}\,\kappa}{2}\,R^{(2)}\,\phi\right), \tag{5.15}$$

where $g(z,\overline{z}\,)$ and $R^{(2)}(z,\overline{z}\,)$ are the metric and curvature of the worldsheet. The topological curvature term in (5.15) provides a deficit term to the central charge c_κ of the free boson field $\phi(z)$,

$$c_\kappa = 1 - 3\kappa^2\,, \tag{5.16}$$

and it also induces a vacuum charge at infinity (the singular point of the metric on the Riemann sphere). In particular, the primary field $\mathrm{e}^{\,\mathrm{i}\,q\phi(z)}$ has dimension

$$\Delta_{q,\kappa} = \frac{q}{2}\left(q-\kappa\right). \tag{5.17}$$

What fixes κ here, and thereby lifts the ambiguity, is the charge conjugation symmetry $\psi^{10}(z)\mapsto -\psi^{10}(z)$ of the NSR model (5.10), which interchanges the two Dirac fields $\psi^\pm(z)$ and hence acts on the free boson field as $\phi(z)\mapsto -\phi(z)$. This symmetry implies that $\kappa=0$ in (5.14).

Let us now consider the tachyon vertex operators corresponding to the free boson,

$$\Sigma_q(z) = \mathrm{e}^{\,\mathrm{i}\,q\phi(z)}\,, \tag{5.18}$$

which have conformal dimension $\Delta_{q,0} = q^2/2$. In bosonized language the pair of Dirac fermion fields corresponds to the operators (5.18) at $q = \pm 1$, $\psi^\pm(z) = \sqrt{2}\,\Sigma_{\pm 1}(z)$. On the other hand, the operators (5.18) at $q = \pm\frac{1}{2}$ introduce a branch cut in the field $\psi^{10}(z)$. To see this, we note the standard free field formula for multi-point correlators of tachyon vertex operators,

$$\begin{aligned} \langle 0|\Sigma_{q_1}(z_1)\cdots\Sigma_{q_n}(z_n)|0\rangle &= \prod_{k=1}^n\prod_{l=1}^n \mathrm{e}^{\,-q_k q_l\langle 0|\phi(z_k)\,\phi(z_l)|0\rangle/2} \\ &= \Lambda^{\left(\sum_l q_l\right)^2/2}\prod_{k<l}(z_k-z_l)^{q_k q_l}\,, \end{aligned} \tag{5.19}$$

where we have regulated the coincidence limit of the two-point function (5.13) by the short-distance cutoff $\Lambda \to 0^+$. In particular, the correlator (5.19) vanishes unless

$$\sum_{l=1}^n q_l = 0\,, \tag{5.20}$$

which is a consequence of the continuous U(1) symmetry generated by the current (5.11) which acts by global translations of the fields $\phi(z_l)$. From the general result (5.19) we may infer the three-point correlation functions

$$\langle 0|\Sigma_{\pm 1/2}(z_1)\,\Sigma_{\pm 1/2}(z_2)\,\Sigma_{\mp 1}(z_3)|0\rangle = \frac{(z_1-z_2)^{1/4}}{\sqrt{(z_1-z_3)(z_2-z_3)}}\,. \tag{5.21}$$

The correlator (5.21) has square root branch points at $z_3 = z_1$ and $z_3 = z_2$. This implies that the elementary fermion fields $\psi^\pm(z_3)$ are double-valued in the fields of the operators $\Sigma_{\mp 1/2}(z_1)$, respectively.

It follows that the spin operators for the recoil problem are given by

$$\Sigma^+_{1/8}(z) = \sqrt{2}\,\cos\frac{\phi(z)}{2} \tag{5.22}$$

and they have weight $\Delta = \Delta_{\pm 1/2} = 1/8$. They create branch cuts in the fermionic fields

$$\psi^{10}(z) = \sqrt{2}\,\sin\phi(z)\,. \tag{5.23}$$

Note that the spin operators need only be inserted at the origin $z = 0$, because it is there that they are required to change the boundary conditions on the fermion fields. These operators are all understood as acting on the NS vacuum state $|0\rangle$, thereby creating highest weight states in the Ramond sector. The spin fields $\Sigma^\pm_{1/8}(0)$ may be extended to operators $\Sigma^\pm_{1/8}(z)$ in the neighbourhood of $\psi^{10}(z)$ via application of the boson translation operator $\mathrm{e}^{\,z\,\partial_z} = \mathrm{e}^{\,z\,L_{-1}}$.

Using the operator product expansions

$$\Sigma_q(z)\,\Sigma_{q'}(w) = (z-w)^{qq'}\,\Sigma_{q+q'}(w)\Big(1 + \mathrm{i}\,q\,(z-w)\,\partial_w\phi(w)\Big) + \cdots\,, \qquad qq' < -1 \tag{5.24}$$

and (5.4), it is straightforward to check that the term of order $(z-w)^{-3/2}$ in the operator product $G(z)\,\Sigma^{+}_{1/8}(w)$ vanishes, and hence that

$$\Sigma^{-}_{1/8}(z)=0\,. \tag{5.25}$$

This means that the spin field $\Sigma(z)=\Sigma^{+}_{1/8}(z)$ corresponds to the supersymmetric ground state $|1/8\rangle_{\rm R}$ in the Ramond sector of the system, associated with superconformal central charge $\hat{c}=2$. By using the selection rule (5.20) and the factorization of bosonic and fermionic correlation functions in the free superconformal field theory determined by (5.1), it is straightforward to verify both the NS two-point functions (3.4)–(3.6) and the spin-spin two-point function as normalized in (3.14). The central charge $\hat{c}=2$ is the one pertinent to the recoil operators because in the bosonized representation they only refer to two of the ten superconformal fields present in the total action (5.1).

Using (5.24) one can also easily derive the excited logarithmic spin operators of dimension $\Delta_{C_\epsilon}+1/8$, which along with (2.22) and (5.5) yields

$$\begin{aligned}\widetilde{\Sigma}_{C_\epsilon}(z) &= {\rm i}\,\epsilon\,C_\epsilon(z)\otimes\sin\frac{\phi(z)}{2}\,,\\ \widetilde{\Sigma}_{D_\epsilon}(z) &= {\rm i}\left(\epsilon\,D_\epsilon(z)-\frac{2}{\epsilon}\,C_\epsilon(z)\right)\otimes\sin\frac{\phi(z)}{2}\,.\end{aligned} \tag{5.26}$$

The corresponding logarithmic operator product expansions (2.23) and (2.24) are straightforward consequences of the factorization of the bosonic and fermionic sectors in the recoil problem. Because of this same factorization property, all of the two-point correlation functions of section 3.2 may be easily derived. The basic identities are given by (5.21) and the four-point function

$$\begin{aligned}\langle 0|\Sigma(z_1)\,\psi^{10}(z)\,\psi^{10}(w)\,\Sigma(z_2)|0\rangle &= \frac{1}{2(z_1-z_2)^{1/4}\,(z-w)}\times\\ &\quad\times\left(\sqrt{\frac{(z_1-z)(w-z_2)}{(z_1-w)(z-z_2)}}+\sqrt{\frac{(z_1-w)(z-z_2)}{(z_1-z)(w-z_2)}}\right), \end{aligned}\tag{5.27}$$

where we have again used the selection rule (5.20). Thus, by using bosonization techniques it is straightforward to describe the $N=1$ supersymmetric extension of the logarithmic operators of the recoil problem in both the NS and R sectors of the worldsheet superconformal field theory.

5.3 Fermionic vertex operators for the Recoil problem

As a simple application of the above formalism, we will now construct the appropriate spacetime vertex operators which create recoil states of the D-branes. The crucial point is that one can now build states that are consistent with the target space supersymmetry of type II superstring theory, which thereby completes the program of constructing recoil operators in string theory. Spacetime supersymmetry necessitates vertex operators which describe the excitations of fermionic states in target space. Such supersymmetric operators were constructed in [16] from a target space perspective. Here we shall construct fermionic states for the recoil problem from a worldsheet perspective by using appropriate combinations of the spin operators (5.18). We have already seen how this arises above, in that the Ramond state $G_0|1/8\rangle_{\rm R}$ is a null vector and one recovers a single logarithmic superconformal algebra among the physical states, as in the NS sector. This construction relies heavily on the euclidean signature of the spacetime, and yields states that transform in an appropriate spinor representation of the euclidean group.

The recoil operators (5.5) are all built as appropriate superpositions of the off-shell tachyon vertex operators $e^{\,{\rm i}\,qx^{10}(z)}$. It is well-known how to construct the boson and fermion emission operators which create corresponding tachyon ground states from the NS vacuum state $|0\rangle$ [29]. In the bosonic sector the vertex operator is $[G_r,\, e^{\,{\rm i}\,qx^{10}(z)}] = q\ e^{\,{\rm i}\,qx^{10}(z)} \otimes \psi^{10}(z)$, where the fermion field $\psi^{10}(z)$ has the periodic mode expansion

$$\psi^{10}(z) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \psi^{10}_{n+1/2}\ z^{-n-1} \tag{5.28}$$

appropriate to the NS sector, with $(\psi^{10}_r)^\dagger = \psi^{10}_{-r}$, $\{\psi^{10}_r, \psi^{10}_s\} = \delta_{r+s,0}$, and $\psi^{10}_{n+1/2}|0\rangle = 0\ \ \forall n \geq 0$. By construction, the corresponding recoil operators are of course just the fermionic operators $\chi_{C_\epsilon}(z)$ and $\chi_{D_\epsilon}(z)$ in (5.5). The emission of a fermion by a spinor u_α is described by the vertex operator $e^{\,{\rm i}\,qx^{10}(z)} \otimes \overline{u}^{\,\alpha}(q)\, \Sigma_\alpha(z)$, where $\alpha = \pm\frac{1}{2}$ are regarded as spinor indices of the two-dimensional euclidean group SO(2) and $u(q)$ is a two-component off-shell Dirac spinor.

The recoil emission vertex operators are therefore given by the chiral superfields

$$\begin{aligned} \mathsf{V}_{C_\epsilon}(\mathsf{z}) &= \Xi_{C_\epsilon}(z) + \theta\, \chi_{C_\epsilon}(z)\,, \\ \mathsf{V}_{D_\epsilon}(\mathsf{z}) &= \Xi_{D_\epsilon}(z) + \theta\, \chi_{D_\epsilon}(z)\,, \end{aligned} \tag{5.29}$$

where the boson emission operators are

$$\begin{aligned} \chi_{C_\epsilon}(z) &= \frac{\epsilon^2}{4\pi} \int\limits_{-\infty}^{\infty} \frac{{\rm d}q}{q - {\rm i}\,\epsilon}\ e^{\,{\rm i}\,qx^{10}(z)} \otimes \psi^{10}(z)\,, \\ \chi_{D_\epsilon}(z) &= -\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{{\rm d}q\ q}{(q - {\rm i}\,\epsilon)^2}\ e^{\,{\rm i}\,qx^{10}(z)} \otimes \psi^{10}(z)\,, \end{aligned} \tag{5.30}$$

while the emission operators for the fermionic recoil states are[2]

$$\Xi_{C_\epsilon}(z) = \frac{\epsilon}{4\pi\,{\rm i}} \int\limits_{-\infty}^{\infty} \frac{{\rm d}q}{q - {\rm i}\,\epsilon}\ e^{\,{\rm i}\,qx^{10}(z)} \otimes \mu(z) \otimes \overline{u}^{\,\alpha}(q)\, \Sigma_\alpha(z)\,,$$

[2]Strictly speaking, the spin operators in these relations should include a non-trivial cocycle [31] for the lattice of charges α in the exponentials of the bosonized representation, which also depend on the fields $\Sigma_\alpha(z)$. The cocycle factor is defined on the weight lattice of the spinor representation of the euclidean group, and it ensures that the vertex has the correct spinor transformation properties. Its inclusion becomes especially important in the generalization of these results to higher-dimensional branes. To avoid clutter in the formulas, we do not write these extra factors explicitly.

$$\Xi_{D_\epsilon}(z) = -\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}q}{(q-\mathrm{i}\,\epsilon)^2}\ \mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)} \otimes \mu(z) \otimes \overline{u}^{\alpha}(q)\,\Sigma_\alpha(z)\,. \tag{5.31}$$

Here $\mu(z)$ is an appropriate auxilliary ghost spin operator of conformal dimension $-1/8$ [28]. For example, it can be taken to be a plane wave $\mu(z) = \mathrm{e}^{\,\mathrm{i}\,k_i x^i(z)}$ in the directions x^i transverse to the (x^9, x^{10}) light cone, with $k^2 = -1/4$. In the physical conformal limit $\epsilon \to 0^+$, the superfields (5.29) then have vanishing superconformal dimension.

The spinor $u(q)$ in (5.31) is not constrained by any on-shell equations such as the Dirac equation which would normally guarantee that the corresponding states respect spacetime supersymmetry. It can be partially restricted by implementing the GSO truncation of the superstring spectrum. The fermion chirality operator Γ acts on the operators (5.18) as

$$\Gamma\,\Sigma_{q+(1-\lambda)/2}(z)\,\Gamma^{-1} = (-1)^{q-\lambda+1}\,\Sigma_{q+(1-\lambda)/2}(z) \tag{5.32}$$

for $q \in \mathbb{Z}$. This is only consistent with the operator product expansions in the combined superconformal field theory including ghost fields, because the action of Γ on the fields of (5.1) alone is not an automorphism of the local algebra of spin fields [28]. Then the action of the chirality operator can be extended to the spin fields with the $\Gamma = 1$ projection giving a local field theory. The chiral $\Gamma = 1$ projection requires that $u_\alpha(q)$ be a right-handed Dirac spinor, after which the operators (5.31) become local fermionic fields. Then the vertex operators (5.29)–(5.31) describe the appropriate supersymmetric states for the recoil problem. The relevant spacetime supersymmetry generator Q_α is given by the contour integral of the fermionic vertex corresponding to the basic tachyon operator $\mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)}$ at zero momentum,

$$Q_\alpha = \oint\limits_{z=0} \frac{\mathrm{d}z}{2\pi}\ \partial_z x^{10}(z) \otimes z^{1/4}\,\mu\left(\frac{1}{z}\right) \otimes \varepsilon_\alpha^{\ \beta}\,\Sigma_\beta(z)\,. \tag{5.33}$$

The integrand of (5.33), which involves the adjoint ghost field $\mu^\dagger(z)$, is a BRST invariant conformal field of dimension 1. From the various operator product expansions above it follows that the supercharge (5.33) relates the two vertices (5.30) and (5.31) through the anticommutators

$$\left\{Q_\alpha\,,\ \mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)} \otimes \mu(z) \otimes \Sigma_\beta(z)\right\} = -\,\mathrm{i}\,\delta_{\alpha\beta}\ \mathrm{e}^{\,\mathrm{i}\,qx^{10}(z)} \otimes \psi^{10}(z)\,. \tag{5.34}$$

Notice, however, that the target space supersymmetry alluded to here refers only to the fields which live on the worldline of the D-particle, or more precisely on the corresponding light-cone. The full target space supersymmetry is of course broken by the motion of the D-brane [16].

6. The $\widehat{c} = -2$ model

By far the best understood example of a logarithmic conformal field theory is the $c = -2$ model. This model has various realizations in terms of ghost fields, symplectic fermions, and twist fields [18, 32]. In this section we shall describe the superconformal extension of

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this logarithmic conformal field theory through the local triplet model [11], and discuss some of the supersymmetric generalizations of the extended algebras that arise in the bosonic case.

6.1 Superconformal symplectic fermions

We begin with the standard superconformal ghost system which has classical action [28]

$$S_{\rm gh} = \frac{1}{2\pi} \int {\rm d}z\ {\rm d}\overline{z}\ {\rm d}\theta\ {\rm d}\overline{\theta}\ \left(\mathsf{b}\, \overline{\mathcal{D}_{\mathsf{z}}}\, \mathsf{c} + \overline{\mathsf{b}}\, \mathcal{D}_{\mathsf{z}} \overline{\mathsf{c}} \right) , \tag{6.1}$$

where the chiral parts of the superfields

$$\begin{aligned} \mathsf{b}(\mathsf{z}) &= \beta(z) + \theta\, b(z)\,, \\ \mathsf{c}(\mathsf{z}) &= c(z) + \theta\, \gamma(z) \end{aligned} \tag{6.2}$$

have superconformal dimensions $1/2$ and 0, respectively. The fermionic (b,c) system has spin $(1,0)$ and central charge -2, while the bosonic (β,γ) system has spin $(1/2,1/2)$ and central charge -1. The superconformal central charge of the combined system is therefore $\hat{c} = -2$. The non-vanishing Neveu-Schwarz two-point functions are

$$\Big\langle \mathsf{b}(\mathsf{z}_1)\, \mathsf{c}(\mathsf{z}_2) \Big\rangle_{\rm NS} = \Big\langle \mathsf{c}(\mathsf{z}_1)\, \mathsf{b}(\mathsf{z}_2) \Big\rangle_{\rm NS} = \frac{\theta_{12}}{\mathsf{z}_{12}}\,, \tag{6.3}$$

and the chiral super energy-momentum tensor is given by

$$\mathsf{T}(\mathsf{z}) = \frac{1}{2}\, \mathcal{D}_{\mathsf{z}}\mathsf{c}(\mathsf{z})\, \mathcal{D}_{\mathsf{z}}\mathsf{b}(\mathsf{z}) - \frac{1}{2}\, \partial_z \mathsf{c}(\mathsf{z})\, \mathsf{b}(\mathsf{z})\,. \tag{6.4}$$

Logarithmic behaviour in the bosonic part of this model is well-known to arise from extending the normal fermionic fields (b,c) to fields with extra zero modes included [18]. This introduces the chiral, two-component scalar symplectic fermion field $\chi^\pm(z)$ through

$$\begin{aligned} b(z) &= \partial_z \chi^-(z)\,, \\ c(z) &= \chi^+(z)\,. \end{aligned} \tag{6.5}$$

The action then has a global $\mathrm{SL}(2,\mathbb{R})$ symmetry acting by rotations of $\chi^\pm$. The Grassmann fields $\chi^\pm(z)$ have mode expansions

$$\chi^\pm(z) = \chi_0^\pm + \tilde{\chi}_0^\pm\, \ln z + \sum_{n\neq 0} \frac{\chi_n^\pm}{n}\ z^{-n} \tag{6.6}$$

with the non-vanishing anticommutation relations

$$\begin{aligned} \left\{ \chi_n^\pm\,,\, \chi_m^\mp \right\} &= \mp\, n\, \delta_{n+m,0}\,, \\ \left\{ \chi_0^\pm\,,\, \tilde{\chi}_0^\mp \right\} &= \pm\, 1 \end{aligned} \tag{6.7}$$

and operator product expansions

$$\chi^+(z)\, \chi^-(w) = -\chi^-(z)\, \chi^+(w) = \ln(z-w) + \cdots\,. \tag{6.8}$$

The symplectic, SL$(2,\mathbb{C})$-invariant Fock vacuum $|0\rangle$ is defined by $\tilde{\chi}_0^{\pm}|0\rangle = \chi_n^{\pm}|0\rangle = 0 \quad \forall n > 0$. The symplectic fermion differs from the fermionic ghost system through the treatment of the $b(z)$ zero mode. The ghost system may be identified with a non-logarithmic subsector of the symplectic fermion model, whose chiral algebra $\mathcal{C}_\chi$ is defined by including the extra zero mode χ_0^- in out-states of correlation functions as

$$\begin{aligned}\Big\langle b(z_1)\cdots b(z_n)\, c(w_1)\cdots c(w_m)\Big\rangle_{\mathcal{C}_\chi} &\equiv \langle\chi_0^-|\partial_{z_1}\chi^-(z_1)\cdots\partial_{z_n}\chi^-(z_n)\,\chi^+(w_1)\cdots\chi^+(w_m)|0\rangle \\ &= \delta_{n+1,m}\prod_{i<i'}(z_i-z_{i'})\prod_{j<j'}(w_j-w_{j'})\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{z_i-w_j}\,,\end{aligned} \tag{6.9}$$

where $|\chi_0^{\pm}\rangle = \chi_0^{\pm}|0\rangle$.

There are several consequences of the presence of these extra non-trivial zero modes. First of all, they couple the holomorphic and antiholomorphic parts of the theory and thereby create a full non-chiral algebra [11]. Nevertheless, we shall continue to concentrate on the chiral sector of the symplectic fermion field theory. Secondly, they are directly responsible for the logarithmic structure of the theory. Because of (6.9), in $\mathcal{C}_\chi$ the vacuum state has zero norm, $\langle 0|0\rangle = 0$, while $\langle 0|\chi^-(z)\,\chi^+(z)|0\rangle = 1$. The field

$$\omega(z) = \chi^-(z)\,\chi^+(z) \tag{6.10}$$

has the operator product expansions

$$\begin{aligned}T(z)\,\omega(w) &= \frac{1}{(z-w)^2} + \frac{1}{z-w}\,\partial_w\omega(w) + \cdots\,, \\ \omega(z)\,\omega(w) &= -\ln^2(z-w) - 2\ln(z-w)\,\omega(w) + \cdots\,,\end{aligned} \tag{6.11}$$

and hence the two-point function

$$\langle 0|\omega(z)\,\omega(w)|0\rangle = -2\ln(z-w)\,. \tag{6.12}$$

It follows that the field $\omega(z)$ is the logarithmic partner of the identity operator I, i.e. the pair of operators $C(z) = I$, $D(z) = \omega(z)$ generate a logarithmic conformal algebra with

$$\begin{aligned}\Delta_I &= 0\,, \\ b &= 1\,, \\ d &= 0\,.\end{aligned} \tag{6.13}$$

In terms of highest weight representations, there is another, degenerate (non-invariant) vacuum state $|\omega\rangle = \omega(0)|0\rangle$ which is conjugate to the SL$(2,\mathbb{C})$-invariant one $|0\rangle$, with $|0\rangle = \tilde{\chi}_0^-\,\tilde{\chi}_0^+|\omega\rangle = L_0|\omega\rangle$.

The superconformal extension of the symplectic fermion system is straightforward. Using the fermionic supercurrent in (6.4) we easily derive the superpartners of the scalar Grassmann fields $\chi^{\pm}(z)$ to be the bosonic spin $\frac{1}{2}$ fields $\psi^{\pm}(z)$, where

$$\begin{aligned}\psi^+(z) &= \gamma(z)\,, \\ \psi^-(z) &= -\beta(z)\,.\end{aligned} \tag{6.14}$$

As expected from Bose statistics, there are no special roles played by the zero modes in this sector. Combining these operators into the chiral scalar superfields

$$\mathsf{X}^{\pm}(\mathsf{z}) = \chi^{\pm}(z) \pm \theta\, \psi^{\pm}(z) \tag{6.15}$$

with the non-vanishing operator product expansion

$$\mathsf{X}^{-}(\mathsf{z}_1)\, \mathsf{X}^{+}(\mathsf{z}_2) = -\ln \mathsf{z}_{12} + \cdots\,, \tag{6.16}$$

we may write the superconformal ghost fields as

$$\begin{aligned} \mathsf{b}(\mathsf{z}) &= \mathcal{D}_{\mathsf{z}} \mathsf{X}^{-}(\mathsf{z})\,, \\ \mathsf{c}(\mathsf{z}) &= \mathsf{X}^{+}(\mathsf{z})\,. \end{aligned} \tag{6.17}$$

The chiral action and super energy-momentum tensor then assume the standard free scalar superfield forms

$$\begin{aligned} S_{\mathrm{gh}} &= \frac{1}{\pi} \int \mathrm{d}^2 z\ \mathrm{d}^2\theta\ \mathcal{D}_{\mathsf{z}} \mathsf{X}^{-}\ \overline{\mathcal{D}_{\mathsf{z}}}\, \mathsf{X}^{+}\,, \\ \mathsf{T}(\mathsf{z}) &= \frac{1}{2} \mathcal{D}_{\mathsf{z}} \mathsf{X}^{+}(\mathsf{z})\, \partial_z \mathsf{X}^{-}(\mathsf{z}) - \frac{1}{2}\, \partial_z \mathsf{X}^{+}(\mathsf{z})\, \mathcal{D}_{\mathsf{z}} \mathsf{X}^{-}(\mathsf{z})\,. \end{aligned} \tag{6.18}$$

The superpartners to the logarithmic operators I and $\omega(z)$ in (6.10) may also be easily determined by using (2.10) and the supercurrent

$$G(z) = \frac{1}{2} \Big(\partial_z \chi^{+}(z) \otimes \psi^{-}(z) + \partial_z \chi^{-}(z) \otimes \psi^{+}(z) \Big) \tag{6.19}$$

to get

$$\begin{aligned} \chi_I(z) &= 0\,, \\ \chi_\omega(z) &= \chi^{+}(z) \otimes \psi^{-}(z) + \chi^{-}(z) \otimes \psi^{+}(z)\,. \end{aligned} \tag{6.20}$$

The operator product expansions (2.9) are straightforward to derive by using (6.16), (6.18) and factorization of the bosonic and fermionic sectors of the free field theory. The NS correlation functions (3.4)–(3.6) are likewise straightforward consequences of factorization and the two-point functions (6.3). Note that $\chi_\omega(z)$ is an ordinary primary field of the Virasoro algebra of weight $\frac{1}{2}$ which corresponds to a supersymmetric state in the NS sector. This is a consequence of the fact that its partner $\chi_C(z)$ in this case is (trivially) a null field, as we generally anticipated in section 4 for the case of zero dimension superconformal logarithmic operators.

6.2 Spin fields

To deal with the R sector of the symplectic fermion superconformal field theory, we proceed via bosonization in complete analogy with section 5.2. Again, there is the conserved U(1) ghost number current (5.11). The bosonization proceeds in exactly the same way as before, except for some crucial sign changes owing to the Bose statistics of the fields $\psi^{\pm}(z)$ in the present case. The bosonization of the U(1) current has the same form as in (5.12), but the chiral scalar field $\phi(z)$ is now non-compact and so has kinetic energy of opposite sign to that of section 5.2, i.e. its two-point function is now

$$\langle 0|\phi(z)\,\phi(w)|0\rangle = \ln(z-w)\,. \tag{6.21}$$

As a consequence, the central extension of the U(1) current algebra is now equal to -1. Due to ghost conjugation symmetry, the Sugawara energy-momentum tensor is given by (5.14) with $\kappa = 0$ and an overall sign change. The tachyon vertex operators (5.18) now have conformal dimension $-\Delta_{q,0} = -q^2/2$, and the relations (5.19) and (5.24) are modified to

$$\begin{aligned}\Sigma_q(z)\,\Sigma_{q'}(w) &= (z-w)^{-qq'}\,\Sigma_{q+q'}(w)\Big(1+\mathrm{i}\,q\,(z-w)\,\partial_w\phi(w)\Big)+\cdots\,, \qquad qq'>1\,,\\ \langle 0|\prod_{l=1}^{n}\Sigma_{q_l}(z_l)|0\rangle &= \delta_{q_1+\ldots+q_n\,,\,0}\,\prod_{k<l}(z_k-z_l)^{-q_kq_l}\,.\end{aligned} \tag{6.22}$$

However, the actual bosonization of the fields $\psi^\pm(z)$ is a bit more involved, because in the present case the U(1) current algebra does not yield their complete dynamics [28]. By Bose statistics, the central charge of the conformal algebra generated by the Sugarawa energy-momentum tensor $(1/8)\,j(z)\,j(z)$ is $c=1$. To get the required central charge -1, we need to add the generators of a conformal algebra with $c=-2$. This is also clear from the fact that the soliton fields $\mathrm{e}^{\pm\mathrm{i}\,\phi(z)}$ are fermionic, while $\psi^\pm(z)$ are bosonic. To this end, we introduce an auxilliary $c=-2$ ghost system (η,ξ), which comprises a pair of conjugate free fermion fields of conformal weights $(1,0)$ with the non-vanishing two-point functions

$$\langle 0|\eta(z)\,\xi(w)|0\rangle = \langle 0|\xi(z)\,\eta(w)|0\rangle = \frac{1}{z-w}\,. \tag{6.23}$$

These auxilliary fields commute with the original $c=-2$ (b,c) system and also with the scalar field $\phi(z)$. Thus, the Ramond sector of the symplectic fermion superconformal field theory involves not only a non-compact U(1) current algebra, but also a fermionic $c=-2$ ghost system (η,ξ).

The Bose fields $\psi^\pm(z)$ can now each be written in terms of a product of two fermionic fields as

$$\begin{aligned}\psi^+(z) &= \sqrt{2}\;\mathrm{e}^{\,\mathrm{i}\,\phi(z)}\otimes\eta(z)\,,\\ \psi^-(z) &= -\sqrt{2}\;\mathrm{e}^{-\mathrm{i}\,\phi(z)}\otimes\partial_z\xi(z)\,.\end{aligned} \tag{6.24}$$

Note that the zero mode of $\xi(z)$ does not contribute and so the auxilliary ghost system does not yield any additional logarithmic behaviour. In fact, the superconformal symplectic fermion field theory is only defined on the subalgebra $\mathcal{C}_\psi$ of the full chiral algebra of the (η,ξ) system which is generated by the fields $\eta(z)$ and $\partial_z\xi(z)$ [18, 28]. In $\mathcal{C}_\psi$ it is consistent to fix the zero-mode of the η field to 0, i.e. $\oint_{z=0}\mathrm{d}z\;\eta(z)=0$. This lifts the two-fold degeneracy of the vacuum state which would otherwise lead to extra logarithmic behaviour. The operator products of the chiral algebra $\mathcal{C}_\psi$ are then obtained by defining its correlation functions as

$$\Big\langle \eta(z_1)\cdots\eta(z_n)\,\partial_{w_1}\xi(w_1)\cdots\partial_{w_m}\xi(w_m)\Big\rangle_{\mathcal{C}_\psi} \equiv \langle\xi_0|\eta(z_1)\cdots\eta(z_n)\,\partial_{w_1}\xi(w_1)\cdots\partial_{w_n}\xi(w_m)|0\rangle\,, \tag{6.25}$$

where $|\xi_0\rangle = (1/2\pi\,\mathrm{i}\,)\,\oint_{z=0}\mathrm{d}z\;\xi(z)|0\rangle$.

As before, we can easily show that the operators $\Sigma_{\mp 1/2}(z)$ create branch cuts in the fields $\psi^{\pm}(z)$. However, because of the auxilliary $c=-2$ $(\eta,\partial_z\xi)$ system that is present, there is no longer a unique spin field [18, 32, 33], but rather an infinite family of them labelled by a continuous parameter $\lambda\in[0,1]$ which interpolates between Neveu-Schwarz and Ramond boundary conditions on the fermion fields $\eta(z)$ and $\xi(z)$. This possibility arises because the conserved U(1) ghost number current $\eta(z)\,\xi(z)$ of the (η,ξ) system can itself be bosonized, yielding twisted sectors of the corresponding Hilbert space. In $\mathcal{C}_\psi$, the bosonization formulas are given by

$$\begin{aligned}\eta(z) &= \frac{1}{\sqrt{2}}\ \mathrm{e}^{\,-\,\mathrm{i}\,\varphi(z)}\,,\\ \partial_z\xi(z) &= \frac{1}{\sqrt{2}}\ \mathrm{e}^{\,2\,\mathrm{i}\,\varphi(z)}\,,\\ \eta(z)\,\xi(z) &= \frac{\mathrm{i}}{4}\,\partial_z\varphi(z)\,,\end{aligned}\tag{6.26}$$

where $\varphi(z)$ is a compact Coulomb gas field of central charge deficit $\kappa=1$. The twist fields in the λ-twisted sector of the (η,ξ) Hilbert space are defined by

$$\mu_\lambda(z) = \mathrm{e}^{\,\mathrm{i}\,\lambda\varphi(z)/2}\,,\tag{6.27}$$

and they generate the local monodromy conditions

$$\begin{aligned}\eta(z)\,\mu_\lambda(w) &= \frac{1}{\sqrt{2}}\Bigl(z-w\Bigr)^{-\lambda/2}\ \mu_{\lambda-2}(w)+\cdots\,,\\ \partial_z\xi(z)\,\mu_\lambda(w) &= \frac{1}{\sqrt{2}}\Bigl(z-w\Bigr)^{\lambda}\ \mu_{\lambda+4}(w)+\cdots\,.\end{aligned}\tag{6.28}$$

From (6.22) and (6.28) it follows that the spin operators which generate $\mathbb{Z}_2$-twists in the fields (6.24) are given by

$$\Sigma^{(\lambda)}(z)=\Sigma^{+}_{-1/8}(z)=2\,\cos\Bigl(\lambda-1\Bigr)\,\frac{\phi(z)}{2}\otimes\mu_\lambda(z)\,.\tag{6.29}$$

For each λ, they have weight

$$\Delta=-\Delta_{(\lambda-1)/2\,,\,0}+\Delta_{\lambda/2\,,\,1}=-\frac{1}{8}\tag{6.30}$$

and therefore correspond to the Ramond ground state $|\hat c/16\rangle_{\rm R}$ for $\hat c=-2$. Thus, in contrast to the model of the previous section, the superconformal symplectic fermion system possesses an infinite family of supersymmetric ground states which are labelled by their U(1) charge $1-\lambda$ in the non-compact current algebra generated by (5.11). This charge partly labels the inequivalent representations of the underlying superconformal algebra. Using (2.22), (6.20), (6.22) and (6.28), we may easily compute the corresponding family of excited logarithmic spin fields of dimension $-1/8$ to be

$$\begin{aligned}\widetilde{\Sigma}{}^{(\lambda)}_I(z) &= 0\,,\\ \widetilde{\Sigma}{}^{(\lambda)}_\omega(z) &= \chi^-(z)\otimes\Sigma_{-(\lambda-3)/2}(z)\otimes\mu_{\lambda-2}(z)-\chi^+(z)\otimes\Sigma_{(\lambda-3)/2}(z)\otimes z^{3\lambda/2}\,\mu_{\lambda+4}(z)\,.\end{aligned}\tag{6.31}$$

Again, the complete logarithmic superconformal algebra of sections 2 and 3 may be easily verified explicitly in the present case by using factorization of the various free field sectors and the correlation functions given above.

6.3 The $sl(2,\mathbb{R})$ symmetry algebra

The generators of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the ordinary symplectic fermion action are given by the conformal dimension 1 primary currents [18]

$$\begin{aligned} J^0(z) &= \frac{1}{3}\Big(\chi^+(z)\,\partial_z\chi^-(z)+\chi^-(z)\,\partial_z\chi^+(z)\Big)\,, \\ J^\pm(z) &= \frac{2}{3}\,\chi^\pm(z)\,\partial_z\chi^\pm(z)\,. \end{aligned} \tag{6.32}$$

By Fermi statistics, these are the complete set of bosonic primary fields with $\Delta = 1$. Together with the logarithmic operator $\omega(z)$, they generate a logarithmic extension of an $sl(2,\mathbb{R})$ Kac-Moody type symmetry algebra [19], which is very similar to an ordinary affine Lie algebra. The relations are given by the operator product expansions (6.11) and

$$\begin{aligned} J^a(z)\,J^b(w) &= \frac{2}{9}\,g^{ab}\left(\frac{\ln(z-w)+\omega(w)+1}{(z-w)^2}+\frac{1/2}{z-w}\,\partial_w\omega(w)\right)+\frac{f^{ab}{}_c}{z-w}\,J^c(w)+\cdots\,, \\ J^a(z)\,\omega(w) &= \ln(z-w)\,J^a(w)+\cdots\,, \end{aligned} \tag{6.33}$$

where g^{ab} and $f^{ab}{}_c$ are the metric and structure constants of the $sl(2,\mathbb{R})$ Lie algebra with, in the chosen basis, the non-vanishing components $g^{+-}=-2g^{00}=-2$ and $f^{+-}{}_3=\mp 2f^{3\pm}{}_\pm=-2$. Up to the $\ln(z-w)$ terms and the appearence of the logarithmic operator (6.10), the relations (6.33) are just those of a standard $sl(2,\mathbb{R})$ affine Lie algebra. This algebra is related to the W-algebra of type $W(2,3^3)$ that arises from combining the $sl(2,\mathbb{R})$ and Virasoro algebras and is normally used to classify the triplet theory [11, 18]. The corresponding generators are obtained from (6.32) by the substitutions $\chi^\pm(z)\mapsto\partial_z\chi^\pm(z)$ [18]. The fields $J^a(z)$ then become the W_3-triplet and the logarithmic operator $\omega(z)$ becomes the bosonic energy-momentum tensor $T(z)$. This eliminates the logarithms in (6.33) (effectively replacing $\ln z$ with $1/z^2$).

As a simple application of the above formalism, let us now examine the supersymmetric enhancement of this logarithmically extended symmetry algebra. As is usual in a supersymmetric current algebra, we regard the bosonic sector current $J^a(z)$ as the highest component of a weight $\frac{1}{2}$ holomorphic superfield

$$\mathsf{J}^a(\mathsf{z}) = j^a(z)+\theta\,J^a(z)\,. \tag{6.34}$$

Using the supercurrent (6.19), the superpartners $j^a(z)$ with $J^a(z)=\chi_{j^a}(z)$ of the currents (6.32) are easily calculated to be the dimension $\frac{1}{2}$ primary fields

$$\begin{aligned} j^0(z) &= \frac{1}{3}\Big(\chi^-(z)\otimes\psi^+(z)-\chi^+(z)\otimes\psi^-(z)\Big)\,, \\ j^\pm(z) &= \pm\frac{2}{3}\,\chi^\pm(z)\otimes\psi^\pm(z)\,. \end{aligned} \tag{6.35}$$

Together with $\omega(z)$, its superpartner $\chi_\omega(z)$ in (6.20), and the bosonic $\mathrm{U}(1)$ ghost number current (5.11), the fermionic currents (6.35) generate an algebra whose non-vanishing operator product expansions are given by (6.11) and

$$j^a(z)\,j^b(z) = \frac{2}{9}\,g^{ab}\,\frac{\ln(z-w)+\omega(w)+1}{z-w}+\frac{2}{9}\,d^{ab}\,\ln(z-w)\,j(w)+\cdots\,,$$

$$
\begin{aligned}
j^a(z)\,\omega(w) &= \ln(z-w)\,j^a(w)+\cdots\,,\\
j^0(z)\,j(w) &= \frac{1/3}{z-w}\,\chi_\omega(w)+\cdots\,,\\
j^\pm(z)\,j(w) &= \pm\,\frac{1}{z-w}\,j^\pm(w)+\cdots\,,\\
j(z)\,j(w) &= -\frac{1}{(z-w)^2}+\cdots\,,\\
j^0(z)\,\chi_\omega(w) &= -\frac{2}{3}\ln(z-w)\,j(w)+\cdots\,,\\
\omega(z)\,\chi_\omega(w) &= \ln(z-w)\,\chi_\omega(w)+\cdots\,,\\
j(z)\,\chi_\omega(w) &= -\frac{3}{z-w}\,j^0(w)+\cdots\,,\\
\chi_\omega(z)\,\chi_\omega(w) &= \frac{2\,\omega(w)-2\ln(z-w)}{z-w}+\cdots\,,
\end{aligned}
\tag{6.36}
$$

where d^{ab} is an $sl(2,\mathbb{R})$ tensor whose only non-vanishing components, in the given basis, are $d^{\pm\mp}=\pm 1$. In addition, there are mixed correlations between these operators and the original currents (6.32) given by the non-vanishing operator product expansions

$$
\begin{aligned}
J^a(z)\,j^b(w) &= \frac{1/3}{z-w}\left(f^{ab}{}_c\,j^c(w)+\frac{g^{ab}}{3}\,\chi_\omega(w)\right)+\cdots\,,\\
J^a(z)\,\chi_\omega(w) &= \frac{1}{z-w}\,j^a(w)+\cdots\,.
\end{aligned}
\tag{6.37}
$$

Again, up to the $\ln(z-w)$ terms and the appearence of the logarithmic superpartner $\chi_\omega(z)$, these are just the usual relations of the $N=1$ superconformal extension of an $sl(2,\mathbb{R})$ Kac-Moody algebra.

Thus the currents (6.32) and (6.35) define a logarithmic extension of the $N=1$ super Kac-Moody symmetry algebra of the superconformal symplectic fermion model. In the bosonic case, the extended W-algebra determines the irreducible representations of the triplet theory and furnishes it with an appropriate definition of rationality [11]. It is therefore the building block of the $c=-2$ model. Under the extension $\chi^\pm(z)\mapsto\partial_z\chi^\pm(z)$ (with $\psi^\pm(z)$ unchanged), the operator $\chi_\omega(z)$ becomes the fermionic supercurrent (6.19), consistently with the transformations of the bosonic sector and the operator product expansions (6.36), (6.37). We may thereby conclude that an $N=1$ superconformal extension of the W-algebra of type $W(2,3^3)$ yields the algebraic characterization of the $\hat{c}=-2$, superconformal symplectic fermion model. This enhancement of the W-symmetry in the supersymmetric case is reminescent of the direct product extension [34] of the $W_{1+\infty}$ symmetry algebra in the twisted $N=2$ superconformal extension of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset current algebra [35].

6.4 Fusion rules in the $\mathbb{Z}_2$ orbifold model

The original (b,c) ghost system of the $c=-2$ model has its own conserved $\mathrm{U}(1)$ current $c(z)\,b(z)$, and as a consequence it possesses itself both twisted and untwisted sectors. The logarithmic behaviour described above originates from the $\mathbb{Z}_2$-twisted sector of the theory [18], in which the symplectic fermion fields $\chi^{\pm}(z)$ have anti-periodic boundary conditions and half-integer weighted mode expansions (6.6). From the U(1) current it is possible to construct, in the usual way, $\mathbb{Z}_2$ spin operators $\mu(z)$ of dimension $-1/8$ which act on the symplectic fermion fields through the operator products

$$\mu(z)\,\partial_w\chi^{\pm}(w) = \frac{1}{\sqrt{z-w}}\,\nu^{\pm}(w) + \cdots\,, \tag{6.38}$$

where $\nu^{\pm}(z)$ are excited twist fields of dimension 3/8. The twist field $\mu(z)$ corresponds to the unique ground state $|\mu\rangle = \mu(0)|0\rangle$ of weight $-1/8$ in the twisted sector, while $\nu^{\pm}(z)$ correspond to the doublet of excited states $|\nu^{\pm}\rangle = \chi^{\pm}_{-1/2}|\mu\rangle$ of dimension 3/8. In the twisted sector, the zero modes in (6.6) are naturally absent, and the logarithmic operator $\omega(z)$ can be split into two twist fields through the fusion relation [1, 32]

$$\mu(z)\,\mu(w) = (z-w)^{1/4}\Big(\omega(w) + \ln(z-w)\Big) + \cdots\,. \tag{6.39}$$

In the language of highest weight representations, the degenerate vacua of the untwisted sector are given by direct products of twisted sector states as [11]

$$\begin{aligned} |\omega\rangle &= |\mu\rangle\otimes|\mu\rangle\,,\\ |0\rangle &= -\frac{1}{4}\,|\mu\rangle\otimes|\mu\rangle + L_{-1}|\mu\rangle\otimes|\mu\rangle\,. \end{aligned} \tag{6.40}$$

The complete set of fusion relations among the twist fields may be summarized succinctly in terms of appropriately defined product representations of the Virasoro algebra. For any quasi-primary or primary chiral field $O(z)$, let $[O]$ denote the irreducible representation built on the ground state $O(0)|0\rangle$. From (6.38) and (6.39) we then have the non-trivial fusion rules [11]

$$\begin{aligned} [\mu]\times[\mu] &= [\omega]\,,\\ [\mu]\times[\nu^{\pm}] &= [\chi^{\pm}_{-1}\,\omega]\,,\\ [\nu^{\pm}]\times[\nu^{\mp}] &= [\omega]\,, \end{aligned} \tag{6.41}$$

where the right-hand sides of (6.41) are indecomposable representations of the Virasoro algebra generated by the logarithmic pairs of states $(|0\rangle\,,\,|\omega\rangle)$ and $(\chi^{\pm}_{-1}|0\rangle\,,\,\chi^{\pm}_{-1}|\omega\rangle)$.

Let us now describe the appropriate superconformal extension of the $\mathbb{Z}_2$ orbifold conformal field theory. Supersymmetry of the symplectic fermion model severely constrains the possible fields. Namely, whatever twist is imposed on the fields $\chi^{\pm}(z)$, a compensating twist should be put in their superpartners $\psi^{\pm}(z)$ such that the fermionic supercurrent (6.19) is single-valued [26]. As we have seen above, $\mathbb{Z}_2$ twists in the superpartner fields $\psi^{\pm}(z)$ are generated by spin operators $\Sigma(z)$ of dimension $-1/8$ through the operator products

$$\Sigma(z)\,\psi^{\pm}(w) = \frac{1}{\sqrt{z-w}}\,\sigma^{\pm}(w) + \cdots\,, \tag{6.42}$$

where the twist fields $\sigma^{\pm}(z)$ also have weight $-1/8$. The bosonization formulas[3] and (6.42) imply the non-trivial fusion rules

$$\begin{aligned} [\Sigma] \times [\Sigma] &= [I] \,, \\ [\Sigma] \times [\sigma^{\pm}] &= [\psi^{\pm}] \,, \\ [\sigma^{\pm}] \times [\sigma^{\mp}] &= [I] \,. \end{aligned} \tag{6.43}$$

In (6.43) we have used supersymmetry to keep only descendants which are compatible with the operator product expansion convention in (6.39). The right-hand sides of (6.43) are highest-weight Virasoro modules.

From the operator product expansions (6.38) and (6.42) it follows that the lowest dimension spin-twist field $\Omega(z)$ which leaves (6.19) single-valued is given by

$$\Omega(z) = \mu(z) \otimes \Sigma(z) \tag{6.44}$$

and it has conformal dimension $-1/4$. Supersymmetry implies that $\Omega(z)$ must have a superpartner. Since it is the spin-twist field of lowest possible dimension, it must be the lowest component of a superfield

$$\Omega(\mathsf{z}) = \Omega(z) + \theta \, \chi_{\Omega}(z) \,. \tag{6.45}$$

a The weight $1/4$ superpartner $\chi_{\Omega}(z)$ of $\Omega(z)$ is easily found by applying the supercurrent (6.19) to (6.44), and using the operator product expansions (6.38) and (6.42) to get

$$\chi_{\Omega}(z) = \nu^{+}(z) \otimes \sigma^{-}(z) + \nu^{-}(z) \otimes \sigma^{+}(z) \,. \tag{6.46}$$

Using (6.41) and (6.43)–(6.46) we may then compute the fusion relations generated by the spin-twist superfield and we find

$$[\Omega] \times [\Omega] = \Big([\omega] \otimes [I]\Big) \oplus \Big([\chi_{-1}^{+} \, \omega] \otimes [\psi^{-}]\Big) \oplus \Big([\chi_{-1}^{-} \, \omega] \otimes [\psi^{+}]\Big) \,. \tag{6.47}$$

This provides the superconformal extension of the first fusion relation in (6.41) and it agrees with the expressions for the logarithmic superfields obtained in (6.20). It demonstrates explicitly how modules over the superconformal algebra in this case split into indecomposable representations of the ordinary Virasoro algebra. The fusion rule (6.47) thereby yields an alternative description of the superconformal symplectic fermion system in terms of fusion relations, whereby the logarithmic operators are descendants of non-integer weight primary fields and are therefore no longer really required in the theory. The Ramond sector in this language is incorporated by introducing, in the usual way, the appropriate spin operators for the fermionic partners $\chi_{\Omega}(z)$ in (6.46). Along with the construction of the previous subsection, this formalism leads to the representation theoretic classification of this logarithmic superconformal field theory. We will not explore this classification any further in this paper.

[3] As before, the proper bosonization relations must include a non-trivial cocycle on the U(1) charge lattice, which in the present case will also depend on the field $\mu(z)$.

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7. Supersymmetric Wess-Zumino-Witten models

As our final example, in this section we will describe explicitly the appearance and properties of logarithmic conformal algebras in $N = 1$ supersymmetric WZW models [36, 37], including the Ramond sector. For definiteness, we shall concentrate on the compact $su(2)_k$ and non-compact $sl(2,\mathbb{R})_k$ Kac-Moody algebras, but a lot of the discussion can be extended to more general current algebras based on generic simple Lie groups. Our analysis will be facilitated enormously by a Coulomb gas representation of these models [38], which allows for a free field representation of the superconformal dynamics. In particular, in this formalism the fermion sector of the supersymmetric WZW model completely decouples from the bosonic sector, thereby yielding free fermion fields which enable a direct construction of the spin operators in analogy with the other examples studied in this paper. As before, the spin fields are completely determined by the chiral, fermionic current algebra of the superconformal field theory.

7.1 Supersymmetric $su(2)_k$ current algebras

Consider the $N = 1$ supersymmetric extension of a level $k \in \mathbb{Z}_+$ Kac-Moody algebra based on the SU(2) group, which is generated by a triplet of supercurrents $\mathsf{J}^a(\mathsf{z})$ as in (6.34) that are holomorphic, weight 1/2 superfields transforming in the adjoint representation of SU(2). Here $j^a(z)$ are spin 1/2 fermionic currents and $J^a(z)$ are bosonic currents of dimension 1 which generate an ordinary Kac-Moody algebra. The operator product expansions of the super Kac-Moody algebra are given by

$$\mathsf{J}^a(\mathsf{z}_1)\,\mathsf{J}^b(\mathsf{z}_2) = \frac{k\,g^{ab}}{2\mathsf{z}_{12}} + \frac{\theta_{12}}{\mathsf{z}_{12}}\,f^{ab}{}_c\,\mathsf{J}^c(\mathsf{z}_2) + \cdots\,. \tag{7.1}$$

In an orthonormal basis, the metric and structure constants of $su(2)$ are given by $g^{ab} = \delta^{ab}$ and $f^{ab}{}_c = \mathrm{i}\,\varepsilon^{ab}{}_c$. In analogy with the bosonic case, the super energy-momentum tensor may be represented in terms of the Kac-Moody supercurrents via a Sugarawa construction as [36, 37]

$$\mathsf{T}(\mathsf{z}) = \frac{1}{k}\,g_{ab}\,\mathcal{D}_{\mathsf{z}}\mathsf{J}^a(\mathsf{z})\,\mathsf{J}^b(\mathsf{z}) + \frac{2}{3k^2}\,f_{abc}\,\mathsf{J}^a(\mathsf{z})\,\mathsf{J}^b(\mathsf{z})\,\mathsf{J}^c(\mathsf{z})\,, \tag{7.2}$$

and from (7.1) it follows that (7.2) generates a super Virasoro algebra (2.3) of superconformal central charge

$$\hat{c} = \frac{3k-4}{k}\,. \tag{7.3}$$

The bosonic part of the supersymmetric WZW model at level k is just the ordinary WZW model at level $k-2$, and the spectrum of the supersymmetric field theory is the same as that of the WZW model at level $k-2$ plus free Majorana fermion fields in the adjoint representation of the SU(2) group [36, 37]. The fermion decoupling can be seen explicitly in terms of the Kac-Moody currents by defining the modified currents

$$\widehat{J}^a(z) = J^a(z) + J^a_{\mathrm{f}}(z)\,, \tag{7.4}$$

where the fermionic currents $J_{\rm f}^a(z)$ are given in terms of the original fermion currents $j^a(z)$ in (6.34) by

$$J_{\rm f}^a(z)=-\frac{1}{k}\,f^a_{~bc}\,j^b(z)\,j^c(z)\,. \tag{7.5}$$

By using (7.1), one finds that the modified currents have the operator product expansions

$$\begin{aligned}
j^a(z)\,j^b(w) &= \frac{k\,g^{ab}}{2(z-w)}+\cdots\,,\\
j^a(z)\,\widehat{J}^b(w) &= \widehat{J}^a(z)\,j^b(w) \;=\; 0+\cdots\,,\\
\widehat{J}^a(z)\,\widehat{J}^b(w) &= \frac{(k-2)\,g^{ab}}{2(z-w)^2}+\frac{f^{ab}_{~~c}}{z-w}\,\widehat{J}^c(w)+\cdots\,,
\end{aligned} \tag{7.6}$$

from which it follows that the operators $\widehat{J}^a(z)$ generate a bosonic Kac-Moody algebra at level $k-2$ and are independent of the fermionic currents $j^a(z)$. The fields $j^a(z)$ pertain to the supersymmetric partners in the current algebra, which thereby decouple from the rest of the quantum field theory.

In the supersymmetric $\mathrm{SU}(2)_k$ WZW model, the primary superfields

$$\mathsf{V}_{j,m}(\mathsf{z})=V_{j,m}(z)+\theta\,W_{j,m}(z) \tag{7.7}$$

are labelled by the usual angular momentum quantum numbers $j\in\mathbb{Z}_+/2$, $m\in\{-j,-j+1,\dots,j-1,j\}$ of the $su(2)$ Lie algebra [36]. They have superconformal dimension

$$\Delta_j=\frac{j(j+1)}{k} \tag{7.8}$$

and obey

$$\begin{aligned}
\mathsf{J}^3(\mathsf{z}_1)\,\mathsf{V}_{j,m}(\mathsf{z}_2) &= \frac{\theta_{12}}{\mathsf{z}_{12}}\,m\,\mathsf{V}_{j,m}(\mathsf{z}_2)+\cdots\,,\\
\mathsf{J}^\pm(\mathsf{z}_1)\,\mathsf{V}_{j,m}(\mathsf{z}_2) &= \frac{\theta_{12}}{\mathsf{z}_{12}}\Big(j\mp m\Big)\,\mathsf{V}_{j,m\pm1}(\mathsf{z}_2)+\cdots\,,
\end{aligned} \tag{7.9}$$

where $\mathsf{J}^\pm(\mathsf{z})=\mathsf{J}^1(\mathsf{z})\pm{\rm i}\,\mathsf{J}^2(\mathsf{z})$. The mode expansion of (7.2) yields null state constraints which, along with the superconformal Ward identities of section 3.1, leads to the supersymmetric version of the Knizhnik-Zamolodchikov equations [37]

$$\left[\mathcal{D}_{\mathsf{z}_l}-\frac{2}{k}\sum_{l'\neq l}\frac{\theta_{ll'}}{\mathsf{z}_{ll'}}\,g_{ab}\,t^a_{(l)}\otimes t^b_{(l')}\right]\Big\langle\mathsf{V}_{j_1,m_1}(\mathsf{z}_1)\cdots\mathsf{V}_{j_n,m_n}(\mathsf{z}_n)\Big\rangle_{\rm NS}=0\,,\qquad l=1,\dots,n\,, \tag{7.10}$$

where $t^a_{(l)}$ are the generators of SU(2) in the spin-j_l representation with

$$\mathsf{J}^a(\mathsf{z}_1)\,\mathsf{V}_{j_l,m_l}(\mathsf{z}_2)=\frac{\theta_{12}}{\mathsf{z}_{12}}\,t^a_{(l)}\,\mathsf{V}_{j_l,m_l}(\mathsf{z}_2)+\cdots\,. \tag{7.11}$$

In particular, from (7.11) we may infer the Ward identities for the supersymmetric Kac-Moody algebra in the form

$$\Big\langle\mathsf{J}^a(\mathsf{z}_0)\,\mathsf{V}_{j_1,m_1}(\mathsf{z}_1)\cdots\mathsf{V}_{j_n,m_n}(\mathsf{z}_n)\Big\rangle_{\rm NS}=\sum_{l=1}^n\frac{\theta_{0l}}{\mathsf{z}_{0l}}\,t^a_{(l)}\Big\langle\mathsf{V}_{j_1,m_1}(\mathsf{z}_1)\cdots\mathsf{V}_{j_n,m_n}(\mathsf{z}_n)\Big\rangle_{\rm NS}\,. \tag{7.12}$$

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The prototypical example for logarithmic behaviour in this class of superconformal field theories is provided by the spin 1/2 primary superfields at level $k = 2$. The decoupled bosonic sector then admits the operator product expansions [14]

$$V_{\frac{1}{2},\alpha_1}(z_1)\,V_{\frac{1}{2},\alpha_2}(z_2) = \frac{1}{(z_1-z_2)^{3/4}}\Bigg[I_{\alpha_1\alpha_2^\vee} - (z_1-z_2)\,t^a_{\alpha_1\alpha_2^\vee}\times \\ \times\Big(D_a(z_2)+\ln(z_1-z_2)\,C_a(z_2)\Big)\Bigg]+\cdots\,, \quad (7.13)$$

where

$$t^a = \frac{\sigma^a}{2} \quad (7.14)$$

are the spin 1/2 generators (with σ^a the usual 2×2 Pauli matrices) satisfying

$$\Big[t^a\,,\,t^b\Big] = \mathrm{i}\,\varepsilon^{ab}{}_c\,t^c\,, \qquad \mathrm{Tr}\,\Big(t^a\,t^b\Big) = \frac{1}{2}\,\delta^{ab}\,, \quad (7.15)$$

and $\alpha = \pm 1/2$ are the fundamental weights of the chiral SU(2) group with $\alpha^\vee$ denoting the weight conjugate to α. From (7.10) it follows that the operators $C_a(z)$ and $D_a(z)$ have the non-vanishing NS two-point functions [14]

$$\begin{aligned}\Big\langle C_a(z)\,D_b(w)\Big\rangle_{\rm NS} &= \frac{\delta_{ab}}{(z-w)^2}\,,\\ \Big\langle D_a(z)\,D_b(w)\Big\rangle_{\rm NS} &= \frac{2\,\delta_{ab}}{(z-w)^2}\Big(1-\ln 4(z-w)\Big)\,,\end{aligned} \quad (7.16)$$

and we thereby obtain a logarithmic superconformal algebra with

$$\begin{aligned}\Delta_{C_a} &= 1\,,\\ b &= 1\,,\\ d &= 2-4\ln 2\,.\end{aligned} \quad (7.17)$$

The indecomposability of the corresponding affine $su(2)_2$ super Lie algebra representation is implied by the Sugarawa construction (7.2). This logarithmic behaviour arises from the fact, discussed above, that the bosonic sector of the supersymmetric $\mathrm{SU}(2)_2$ WZW model is just the ordinary SU(2) WZW model at level $k = 0$, which is a well-known example of a $c = 0$ theory which is both logarithmic and unitary [6, 14], and therefore contains no negative dimension operators. Taking the product of the bosonic logarithmic operators with the corresponding free fermion fields then yields their superpartners, and hence the logarithmic behaviour described above.

At the level of the fusion relations (7.13), one cannot determine the precise form of the logarithmic pair. We will find these operators explicitly later on by another technique. The basic idea is that although the classical WZW action vanishes in the bosonic $\mathrm{SU}(2)_0$ model, the effective *quantum* action is non-vanishing [21] and yields a deformation of the classical energy-momentum tensor which embeds the $c = 0$ theory into a twisted $N = 2$ superconformal algebra [35]. The BRST supercharge of the corresponding topological field theory is the zero mode of one of the fermionic supercurrents of this $N = 2$ algebra, which is used as a deformation field. The BRST operator can be represented in terms of a fermionic screening operator for the WZW model, from which the logarithmic operators may be explicitly constructed.

7.2 Coulomb gas representation

To facilitate our analysis, we shall now construct a free field representation of the current algebra in the superfield formalism [38] and use it to construct highest-weight representations of the supersymmetric $su(2)_k$ algebra in the Fock space of the free fields. For this, we introduce three free, real scalar superfields

$$\Phi^a(\mathsf{z}) = \phi^a(z) - \mathrm{i}^{\,a}\,\theta\,\psi^a(z)\,, \qquad a = 1,2,3\,, \tag{7.18}$$

which have the NS two-point functions

$$\Big\langle \Phi^{\mathsf{a}}(\mathsf{z}_1)\,\Phi^{\mathsf{b}}(\mathsf{z}_2)\Big\rangle_{\rm NS} = (-1)^{\mathsf{a}}\,\delta^{\mathsf{ab}}\,\ln \mathsf{z}_{12}\,. \tag{7.19}$$

For simplicity, here and in the remainder of this paper we shall work in an orthonormal basis of the $su(2)$ Lie algebra.

The superspace currents $\mathsf{J}^a(\mathsf{z})$ can be expressed as

$$\begin{aligned} \mathsf{J}^3(\mathsf{z}) &= \mathrm{i}\,\sqrt{\frac{k}{2}}\,\mathcal{D}_{\mathsf{z}}\Phi^3(\mathsf{z})\,, \\ \mathsf{J}^\pm(\mathsf{z}) &= \sqrt{\frac{k}{2}}\,\Big(\mathcal{D}_{\mathsf{z}}\Phi^2(\mathsf{z}) \mp \mathcal{D}_{\mathsf{z}}\Phi^1(\mathsf{z})\Big)\;\mathrm{e}^{\pm\mathrm{i}\,\sqrt{2/\mathsf{k}}\,(\Phi^2(\mathsf{z})+\Phi^3(\mathsf{z}))}\,. \end{aligned} \tag{7.20}$$

The Sugarawa operator (7.2) can also be expressed in terms of the superfields (7.18) as

$$\mathsf{T}(\mathsf{z}) = \frac{1}{2}\sum_{a=1}^{3}(-1)^a\,\partial_z\Phi^{\mathsf{a}}(\mathsf{z})\,\mathcal{D}_{\mathsf{z}}\Phi^{\mathsf{a}}(\mathsf{z}) + \frac{\mathrm{i}}{\sqrt{2\mathsf{k}}}\,\partial_{\mathsf{z}}\mathcal{D}_{\mathsf{z}}\Phi^1(\mathsf{z})\,, \tag{7.21}$$

which in component form yields an expression for the bosonic energy-momentum tensor that is familiar from the usual Coulomb gas representation

$$T(z) = \frac{1}{2}\sum_{a=1}^{3}\Big((-1)^a\,\partial_z\phi^a(z)\,\partial_z\phi^a(z) - \psi^a(z)\,\partial_z\psi^a(z)\Big) + \frac{\mathrm{i}}{\sqrt{2k}}\,\partial_z^2\phi^1(z)\,, \tag{7.22}$$

$$G(z) = -\frac{1}{2}\sum_{a=1}^{3}\mathrm{i}^{\,a}\,\partial_z\phi^a(z)\otimes\psi^a(z) + \frac{1}{\sqrt{2k}}\,\partial_z\psi^1(z)\,. \tag{7.23}$$

It is straightforward to check that this Coulomb gas representation correctly reproduces the superconformal central charge (7.3). The last term in (7.22) yields a background charge $-1/\sqrt{k}$ at infinity in the Coulomb gas model for the ϕ^1 field, i.e. a central charge deficit

$$\kappa = \sqrt{\frac{2}{k}} \tag{7.24}$$

in the notation of (5.14), (5.15).

Let us now exhibit explicitly the decoupling of the fermionic partners from the bosonic Kac-Moody algebra at level $k-2$, represented by the operator product expansions (7.6), in the Coulomb gas representation. For this, we need to decouple the fermion fields $\psi^a(z)$ from the scalar fields $\phi^a(z)$ in the expressions for the modified fermionic currents of the previous subsection. We begin by bosonizing the complex fermion fields

$$\psi^{\pm}(z) \equiv \psi^2(z) \mp \mathrm{i}\,\psi^1(z) = \sqrt{2}\ \mathrm{e}^{\pm\mathrm{i}\,\phi^4(z)} \tag{7.25}$$

by means of a fourth scalar boson field $\phi^4(z)$ with the two-point function

$$\langle 0|\phi^4(z)\,\phi^4(w)|0\rangle = -\ln(z-w)\,, \tag{7.26}$$

just as we did in section 5.2. We now define an $\mathrm{SO}(2,1)$ transformation of the four boson fields $\phi^i(z)$, $i=1,\dots,4$ by

$$\begin{aligned}
\varphi^1(z) &= \phi^1(z)\,,\\
\varphi^2(z) &= \sqrt{\frac{2}{k-2}}\,\phi^2(z) + \sqrt{\frac{2}{k-2}}\,\phi^4(z)\,,\\
\varphi^3(z) &= -\frac{2}{\sqrt{k(k-2)}}\,\phi^2(z) + \sqrt{\frac{k-2}{k}}\,\phi^3(z) - \sqrt{\frac{2}{k-2}}\,\phi^4(z)\,,\\
\varphi^4(z) &= \sqrt{\frac{2}{k}}\,\phi^2(z) + \sqrt{\frac{2}{k}}\,\phi^3(z) + \phi^4(z)\,,
\end{aligned} \tag{7.27}$$

with the corresponding two-point functions

$$\langle 0|\varphi^i(z)\,\varphi^j(w)|0\rangle = -(-1)^{\delta_{i,2}}\,\delta^{ij}\,\ln(z-w)\,. \tag{7.28}$$

Note that (7.27) is only defined for $k>2$. With this transformation, the modified currents (7.4) decouple from the boson field $\varphi^4(z)$, and hence the fermion fields, and are given explicitly by

$$\begin{aligned}
\widehat{J}^3(z) &= \mathrm{i}\,\sqrt{\frac{k-2}{2}}\,\partial_z\varphi^3(z)\,,\\
\widehat{J}^{\pm}(z) &= \sqrt{\frac{k-2}{2}}\left(\partial_z\varphi^2(z) \mp \sqrt{\frac{k}{k-2}}\,\partial_z\varphi^1(z)\right)\ \mathrm{e}^{\pm\mathrm{i}\,\sqrt{2/(k-2)}\,(\varphi^2(z)+\varphi^3(z))}\,.
\end{aligned} \tag{7.29}$$

It is straightforward to check directly that the operators (7.29) generate a bosonic $su(2)_{k-2}$ Kac-Moody algebra. Indeed, (7.29) is just the Nemeschansky representation of the ordinary $su(2)$ current algebra at level $k-2$ [39].

On the other hand, we can fermionize back the boson field $\varphi^4(z)$ by introducing new free fermion fields $\tilde{\psi}^a(z)$ through

$$\begin{aligned}
\tilde{\psi}^{\pm}(z) &= \sqrt{2}\ \mathrm{e}^{\pm\mathrm{i}\,\varphi^4(z)}\,,\\
\tilde{\psi}^3(z) &= \psi^3(z)\,,
\end{aligned} \tag{7.30}$$

and express the fermionic currents $j^a(z)$ solely in terms of three free Majorana fermion fields as

$$j^a(z) = \sqrt{\frac{k}{2}}\,\tilde{\psi}^a(z)\,. \tag{7.31}$$

It follows that the supersymmetric $su(2)_k$ affine Lie algebra decomposes into bosonic algebras as $su(2)_{k-2} \oplus su(2)_2$, where $su(2)_2$ is the current algebra associated with the three real fermion fields $\tilde{\psi}^a(z)$. Let us note from (7.29) that the bosonic $su(2)_{k-2}$ currents formally vanish identically at $k = 2$, and the supercurrents can be written as $\mathsf{J}^a(\mathsf{z}) = \tilde{\psi}^a(z) + \theta\, \varepsilon^a{}_{bc}\, \tilde{\psi}^b(z)\, \tilde{\psi}^c(z)$. Thus the minimal level $k = 2$ supersymmetric Kac-Moody algebra is described solely in terms of three Majorana fermion fields, or equivalently by means of a free boson field and a free fermion field, i.e. a single scalar superfield. For $k > 2$ the affine algebra is determined instead by three scalar superfields.

Finally, let us describe how to evaluate correlation functions of primary operators in the above free field representation. The vertex operators for the primary superfields (7.7) can be written in terms of the scalar superfields (7.18) as

$$\mathsf{V}_{j,m}(\mathsf{z}) = \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\left[-j\,\Phi^1(\mathsf{z})+\mathsf{m}\left(\Phi^2(\mathsf{z})+\Phi^3(\mathsf{z})\right)\right]}\,. \tag{7.32}$$

For $k > 2$, the bosonic component of (7.32) is mapped under the $\mathrm{SO}(2,1)$ transformation (7.27) to

$$V_{j,m}(z) = \mathrm{e}^{\,\mathrm{i}\left[-j\,\sqrt{2/k}\,\varphi^1(z)+m\,\sqrt{2/(k-2)}\,\left(\varphi^2(z)+\varphi^3(z)\right)\right]}\,, \tag{7.33}$$

which, as expected, coincides with the primary fields of the bosonic level $k-2$ Kac-Moody algebra for the spin j highest weight representation. The superpartner of (7.33) in this decoupling free field representation may be determined in the NS algebra through the supersymmetry transformation $W_{j,m}(z) = [G_{-1/2}, V_{j,m}(z)]$.

To compute correlation functions of the vertex operators (7.32), we need to insert appropriate Feigin-Fuks operators [40] which commute with the Kac-Moody supercurrents and have vanishing superconformal dimension. As such, their insertion into correlators of primary superfields does not affect their conformal or affine Ward identities, but merely serve to "screen" out extra charges such that the correlators satisfy the appropriate neutrality conditions required of (non-vanishing) singlet solutions to the Knizhnik-Zamolodchikov equations (7.10). Screening operators are not unique, and they are provided by supercontour integrals of the form [41]

$$Q = \oint\limits_{z=0} \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \int \mathrm{d}\theta\ \mathsf{Q}(\mathsf{z}) = \oint\limits_{z=0} \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\ E(z)\,, \tag{7.34}$$

where $\mathsf{Q}(\mathsf{z}) = \rho(z) + \theta\, E(z)$. A screening operator for the compact, $\mathrm{SU}(2)_k$ supersymmetric WZW model may be constructed to conveniently have only Φ^1-charge and is given by

$$\mathsf{Q}(\mathsf{z}) = \mathcal{D}_{\mathsf{z}}\Phi^2(\mathsf{z})\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/\mathsf{k}}\,\Phi^1(\mathsf{z})}\,. \tag{7.35}$$

In terms of the decoupling free field representation (7.27), the weight 1 fermionic component of (7.35) may be written as (up to normalization for $k > 2$)

$$E(z) = \partial_z\varphi^2(z)\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\varphi^1(z)}\,. \tag{7.36}$$

That the scalar field $\varphi^1(z)$ is appropriate for the screening operator here is clear from the representation (7.22) of the energy-momentum tensor, whereby the $\partial_z^2\varphi^1(z)$ term is responsible for inducing a vacuum charge $\mathrm{e}^{-\mathrm{i}\sqrt{2/k}\,\varphi^1(\infty)}$ (see (5.15)). Indeed, one can check explicitly that the first order pole in the operator product expansion of $E(z)\,\widehat{J}^a(w)$ vanishes.

The insertion of (7.36) into primary state correlators rids us of the background neutralizing constraint $\sum_l q_l = \kappa$ required of the ϕ^1 tachyon vertex operators, which generalizes (5.20) to the case of a non-vanishing central charge deficit κ. To calculate correlation functions explicitly in the Coulomb gas representation, we insert screening charges (7.34) and define vertex operators which are dual to (7.32) by

$$\tilde{\mathsf{V}}_{j,m}(\mathsf{z}) = \mathrm{e}^{\,\mathrm{i}\sqrt{2/k}\left[(j+1)\,\Phi^1(\mathsf{z})+\mathsf{m}\left(\Phi^2(\mathsf{z})+\Phi^3(\mathsf{z})\right)\right]}\,. \tag{7.37}$$

Then the NS correlators in the free field representation are defined by

$$\begin{aligned}\Big\langle \mathsf{V}_{j_1,m_1}(\mathsf{z}_1)\cdots\mathsf{V}_{j_n,m_n}(\mathsf{z}_n)\Big\rangle_{\mathrm{NS}} = \delta_{m_1+\cdots+m_n\,,\,0}\,\langle 0|\mathsf{V}_{j_1,m_1}(\mathsf{z}_1)\cdots\mathsf{V}_{j_{n-1},m_{n-1}}(\mathsf{z}_{n-1})\,\times \\ \times\,\tilde{\mathsf{V}}_{j_n,m_n}(\mathsf{z}_n)\;Q^{j_1+\cdots+j_{n-1}-j_n}|0\rangle\,,\end{aligned} \tag{7.38}$$

where the charge neutrality constraint $\sum_l m_l = 0$ follows from the Ward identity (7.12) for the super affine SU(2) symmetry.

7.3 Emergence of chiral logarithmic operators

We will now proceed to an explicit construction of logarithmic operators in the supersymmetric $\mathrm{SU}(2)_k$ WZW model for appropriate choices of the level $k \in \mathbb{Z}_+$. The construction is motivated by the manner in which logarithmic behaviour arises in fusion relations among the chiral primary superfields (7.32). For instance, at level $k = 2$ for the spin $j' = 1/2$ representation, it follows from (7.13) that the logarithmic operators transform as a conjugate representation of SU(2) and have dimension $\Delta_1 = 2/k = 1$. This suggests that generally, for certain values of k, logarithmic operators arise in the conjugate spin-$2j'$ representation in the Clebsch-Gordan decomposition of the fusion product of two spin-j' primary fields according to the affine SU(2) fusion rules

$$[V_{j'}] \times [V_{j'}] = [I] \oplus [V_1] \oplus [V_2] \oplus \cdots \oplus [V_{2j'}]\,. \tag{7.39}$$

In the following we will show that this is indeed the case for the spin 1/2 representation, which necessitates the level number $k = 2$. For other spins j' we have no general proof that this is the case, but nevertheless logarithmic operators *can* be built for certain k. The precise values of j' and k will be determined by demanding that the logarithmic pair determine a marginal deformation of the original superconformal field theory, and thereby generate an isomorphic superconformal algebra.

We will first present a very simple version of this construction, and then afterwards show more precisely that it amounts to a deformation of the superconformal algebra. For this, we note that logarithmic behaviour in a conformal field theory arises when two (primary) operators, one of negative norm and the other of positive norm, have weights which become degenerate [42]. Their other quantum numbers must also coincide, so that they form a Jordan cell structure with respect to the full, maximally extended chiral symmetry algebra. Motivated by these remarks, we begin with the chiral primary fields (7.33) written in terms of the original scalar fields $\phi^a(z)$,

$$V_{j,m}(z) \,=\, \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\left[-j\,\phi^1(z)+m\left(\phi^2(z)+\phi^3(z)\right)\right]}\,. \tag{7.40}$$

It is important to note that, as follows from (7.22), with respect to the relative signs in the respective kinetic energies, the field $\phi^2(z)$ is time-like, while $\phi^1(z)$ and $\phi^3(z)$ are space-like. Furthermore, $\phi^2(z)$ and $\phi^3(z)$ are free fields, while $\phi^1(z)$ is a Coulomb gas field with central charge deficit (7.24).

We now consider the state with $m = 0$ and *deform* the operator (7.40) by giving the ϕ^2 field a coefficient $\sqrt{2/k}\,(m+\epsilon)$, where $\epsilon \to 0^+$ is an infinitesimally small, but non-zero, number. Thus we define the field

$$V_{j,\epsilon}(z) \,=\, \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\left(-j\,\phi^1(z)+\epsilon\,\phi^2(z)\right)}\,. \tag{7.41}$$

From (7.22) it follows that the operator (7.41) has conformal dimension

$$\Delta_j(\epsilon) = \frac{j(j+1)}{k} - \frac{\epsilon^2}{k}\,, \tag{7.42}$$

where the minus sign in the second term is due to the time-like signature of the free field $\phi^2(z)$. We will only consider those states deformed in this way whose spin j, for a given level k, is subject to the constraint

$$j(j+1) = k\,. \tag{7.43}$$

In that case, the conformal dimension (7.42) deviates from unity by an infinitesimally negative amount, thereby making the operator (7.41) *relevant* in a renormalization group sense. In the limit $\epsilon \to 0^+$, the operator (7.41) thus yields a marginal deformation of the superconformal field theory and an isomorphic superconformal algebra, i.e. because of the time-like signature of the field $\phi^2(z)$, deforming its momentum does *not* produce off-shell quantities.

Before moving on with the construction, let us pause to briefly discuss what the implications of this deformation will be. The unitary and topological (or rational) constraint $k \in \mathbb{Z}_+$ implies that the only spin-j representations which can be used in the ensuing construction, i.e. those which obey (7.43), are the integral ones $j = 1, 2, 3, 4, \ldots$. This is consistent with the unitarity of the corresponding SU(2) representation and the fact that the deformation requires states of magnetic quantum number $m = 0$. It will restrict logarithmic behaviour in supersymmetric $su(2)_k$ current algebras to those with level coefficient $k = 2, 6, 12, 20, \ldots$, respectively. Logarithmic behaviour can occur for other discrete values of k not in this list [43], but in that case the operators will not yield a marginal deformation of the superconformal algebra. Notice that only the case $k = 2$ violates the ground state unitarity condition $2j \leq k-2$, once again illustrating the special features of this level number. Fractional level numbers are also possible [12, 43], but in that case the deformation will not apply. Finally, note that $k \in \mathbb{Z}_-$ implies by (7.43) that also $j \in \mathbb{Z}_-$, leading to non-unitary, continuous representations of the $su(2)_k$ current algebra. Such values are most naturally thought of in terms of non-compact, $sl(2,\mathbb{R})_{-k}$ WZW models, which we shall briefly discuss in section 7.7.

With the restriction (7.43), let us now define the primary operator

$$C_\epsilon(z) = -\,\mathrm{i}\,\epsilon\, V_{j,\epsilon}(z) = -\,\mathrm{i}\,\epsilon\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/j(j+1)}\,\left(-j\,\phi^1(z)+\epsilon\,\phi^2(z)\right)}\ . \tag{7.44}$$

This field has a logarithmic partner $D_\epsilon(z)$ which may be found either by computing the operator product of the energy-momentum tensor (7.22) with (7.44), or by the formal derivative rule [27] $D_\epsilon = \partial C_\epsilon/\partial\Delta_j(\epsilon) = -(k/2\epsilon)\,\partial C_\epsilon/\partial\epsilon$. Using either method we find

$$D_\epsilon(z) = -\sqrt{\frac{j(j+1)}{2}}\ \phi^2(z)\ \ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/j(j+1)}\,\left(-j\,\phi^1(z)+\epsilon\,\phi^2(z)\right)}\ . \tag{7.45}$$

Since the operator (7.45) contains the free scalar field $\phi^2(z)$ itself (and not merely exponentials or derivatives thereof), it has logarithmic correlation functions. In writing (7.45) we have found it convenient to exploit the ambiguity in the definition of the D operator to eliminate the terms of order $1/\epsilon$ which diverge as $\epsilon\to 0^+$ and are due to the particular form of the operator (7.44). These operators then obey the standard logarithmic conformal algebra provided that one uses the free field prescription (7.38) with *connected* correlation functions $\langle O\,O'\rangle_{\rm conn} = \langle O\,O'\rangle - \langle O\rangle\,\langle O'\rangle$ [44], and makes the identification (5.7).[4]

Under the operator-state correspondence, the state created by the "puncture" operator (7.45) is the logarithmic partner of the highest-weight state $|j,0\rangle$ corresponding to the primary field $V_{j,0}(z)$. It is straightforward to see, at least at $k=2$, that (7.44), (7.45) is the logarithmic pair that arises through appropriate fusion products of primary operators, as in (7.13). In terms of the fusion rules (7.39), the condition (7.43) restricts the spin j' used in the operator product according to

$$j'\left(j'+\frac{1}{2}\right) = \frac{k}{4}\,. \tag{7.46}$$

Let $m'\in\{-j',-j'+1,\dots,j'-1,j'\}$. Using (6.22), with $m_1'=m'$, $m_2'=-m'+\epsilon$ and the constraint (7.46) we may compute in the limit $\epsilon\to 0^+$ the operator product expansion

$$\begin{aligned}\mathrm{e}^{\,\mathrm{i}\,m_1'\sqrt{2/k}\,\phi^2(z)}\ \ \mathrm{e}^{\,\mathrm{i}\,m_2'\sqrt{2/k}\,\phi^2(w)} &= (z-w)^{2m'^{\,2}/k}\left(1-\frac{2m'\,\epsilon}{k}\,\ln(z-w)\right)\times\\ &\quad\times\mathrm{e}^{\,\mathrm{i}\,\epsilon\sqrt{2/k}\,\phi^2(w)}+\cdots\,.\end{aligned} \tag{7.47}$$

This illustrates how, by deforming the momentum of the ϕ^2 field, logarithmic singularities will be produced at leading order ϵ in four-point correlation functions of the form

$$F_{j',m'\,;\,l',n'}(z_1,z_2\,;\,w_1,w_2) = \Big\langle V_{j',m'}(z_1)\,V_{j',-m'}(z_2)\,V_{l',n'}(w_1)\,V_{l',-n'}(w_2)\Big\rangle_{\rm NS}\,. \tag{7.48}$$

[4]There are interesting similarities between the deformation introduced here and the logarithmic recoil operators in (5.5). Formally, the x^{10} field is analogous to the ϕ^2 field here. Setting $\epsilon=0^+$ in the Fourier integrals of (5.5) yields formally the Heaviside function $\Theta(x^{10})$, which has vanishing momentum $q=0$ and hence yields a unit weight deformation in the action (5.1). This is similar to setting $m=0$ here. As follows from the expressions for the logarithmic operators which follow, the states are related formally by taking the $k\to\infty$ limit of the WZW model. From (7.43) we then have also $j\to\infty$ with $k=j^2$, and the logarithmic operators appear to map into each other. It would be interesting to understand this formal correspondence better, particularly in light of the analysis of the next subsection 7.4.

Using (7.38) we write (7.48) explicitly as

$$F_{j',m'\,;\,l',n'}(z_1,z_2\,;\,w_1,w_2) = \langle 0|V_{j',m'}(z_1)\,V_{j',-m'}(z_2)\,V_{l',n'}(w_1)\,\tilde{V}_{l',-n'}(w_2)\;Q^{2j'}|0\rangle\,. \quad (7.49)$$

Note that at $k=2$, the decoupled bosonic sector of the theory has vanishing central charge, and (7.34) is the nilpotent BRST operator of the theory [45], i.e. $Q^2=0$. Then (7.49) vanishes unless $j'=\frac{1}{2}$, which is consistent with the condition (7.46). For $k>2$, Q is only nilpotent when acting on appropriate Feigin-Fuks representations.

The logarithms arising in (7.48) are accounted for by the fusion products

$$V_{j',m'}(z)\,V_{j',-m'}(w) = (z-w)^{j'/(2j'+1)}\Big(D_\epsilon(w)+\ln(z-w)\,C_\epsilon(w)\Big)+\cdots\,, \quad (7.50)$$

where the C_ϵ operator arises from (7.47) while the D_ϵ operator arises, at least for $k=2$, from the insertion of the screening operator (7.36). This illustrates how the logarithmic operators constructed above appear as descendant states of non-integer weight primary fields, in the manner described at the beginning of this subsection. Let us stress that while we have only really proven the appearence of (7.45) in this manner for $k=2$, the construction of logarithmic operators above holds for all k obeying the constraint (7.43). In the next subsection 7.4 we shall give another, similar explanation for the appearence of the logarithmic pair (7.44,7.45).

Finally, to construct the superpartners of the logarithmic pair (7.44,7.45), we apply the supercurrent (7.23) to the primary field (7.40) to get its superpartner

$$W_{j,m}(z) = \sqrt{\frac{2}{k}}\;\mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\left[-j\,\phi^1(z)+m\left(\phi^2(z)+\phi^3(z)\right)\right]}\otimes\Big[-j\,\psi^1(z)+m\Big(\mathrm{i}\,\psi^2(z)+\psi^3(z)\Big)\Big]\,. \quad (7.51)$$

The partner of the operator (7.44) is then obtained by setting $m=0$ and shifting the coefficient of the ψ^2 field by ϵ in (7.51) to give

$$\chi_{C_\epsilon}(z) = \mathrm{i}\,\sqrt{\frac{2}{j(j+1)}}\,C_\epsilon(z)\otimes\Big(-j\,\psi^1(z)+\,\mathrm{i}\,\epsilon\,\psi^2(z)\Big)\,. \quad (7.52)$$

Differentiating (7.52) then gives the superpartner of (7.45) as

$$\chi_{D_\epsilon}(z) = \sqrt{\frac{j(j+1)}{2}}\,\left(D_\epsilon(z)+\frac{1}{\epsilon}\,C_\epsilon(z)\right)\otimes\psi^2(z)\,, \quad (7.53)$$

where as before we have removed divergent terms of order $1/\epsilon$. One can now easily verify the NS sector logarthmic superconformal algebra with the parameters (7.17).

7.4 Deforming the current algebra

We shall now relate the deformation above to deformations of the energy-momentum tensor [13] and hence prove invariance of the superconformal field theory with respect to the appearence of such operators. This will provide a better understanding of the origin of logarithmic operators in WZW models. Recall that the logarithmic pair is constructed by deforming the operator (7.40) via a shift in the coefficient of the ϕ^2 field by an infinitesimal

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amount. This clearly deforms the entire $su(2)_k$ current algebra $\mathcal{C}$ of fields of the supersymmetric WZW model. We shall now describe the nature of this deformation and how it leads to the appearence of Jordan cells in a more precise way.

For this, we consider the mode expansion of the weight 1 fermionic screening operator (7.36),

$$E(z) = \frac{Q}{z} + \sum_{n\neq 0} E_n \, z^{-n-1} \,, \tag{7.54}$$

with Q the screening charge (7.34), and the vacuum module conditions $Q|0\rangle = E_n|0\rangle = 0 \ \ \forall n \geq 1$. We then introduce the field $\nabla_E(z)$ defined by

$$\partial_z \nabla_E(z) = E(z) \,, \tag{7.55}$$

which using (7.54) has the mode decomposition

$$\nabla_E(z) = \chi_0 + Q \ln z - \sum_{n\neq 0} \frac{E_n}{n} \, z^{-n} \tag{7.56}$$

with χ_0 an arbitrary fermionic zero mode. The operator product expansion of $\nabla_E(z)$ with any field $O(z)$ of the current algebra is given by

$$\begin{aligned} \nabla_E(z)\, O(w) \;=\; & \ln(z-w) \Big[Q \,,\, O(w) \Big] + \\ & + \sum_{n=1}^{\infty} \frac{1}{n} \, \frac{1}{(z-w)^n} \sum_{l=1}^{n+1} (-1)^{n+l} \binom{n}{l-1} \, w^{n-l+1} \Big[E_{l-1} \,,\, O(w) \Big] + \cdots \,, \end{aligned} \tag{7.57}$$

which along with (5.6) can be used to define a deformation $O^E(z)$ of the field $O(z)$ by

$$O^E(z) = O(z) + \chi_0^\dagger \, \nabla_E(z) \, O(z) \,, \tag{7.58}$$

where $\chi_0^\dagger$ is a Grassmann-valued mode which is conjugate to χ_0,

$$\Big\{ \chi_0 \,,\, \chi_0^\dagger \Big\} = 1 \,. \tag{7.59}$$

The operation $O \mapsto O^E$ defines a graded outer derivation of the corresponding operator product algebra [13].

Since Q is a screening operator for the current algebra, it serves as a BRST charge for the physical state space and $[Q, O(z)] = 0$. Thus the logarithms naturally disappear from (7.57) and (7.58). Since $E(z)$ is a weight 1 primary field, it satisfies

$$\sum_{l=1}^{n} (-1)^{n+l} \binom{n-1}{l-1} \, z^{n-l} \Big[E_{l-1} \,,\, T(z) \Big] = \delta_{n,2} \, E(z) \,, \qquad n > 0 \,, \tag{7.60}$$

and hence the corresponding deformation of the energy-momentum tensor is given by

$$T^E(z) = T(z) + \frac{\chi_0^\dagger}{z} \, E(z) \,. \tag{7.61}$$

This ensures that $T^E(z)$ has the correct operator product to be an energy-momentum tensor, i.e. to generate a Virasoro algebra.

Since logarithms do not appear in the transformed fields (7.58), the algebra of the deformed operators $O^E(z)$ is *isomorphic* to the original current algebra $\mathcal{C}$. In complete analogy with the relationship between the fermionic ghost system and the symplectic fermion in section 6.1, the WZW model is identified with a subsector of a larger conformal field theory whose chiral algebra $\mathcal{C}_E$ of fields contains the operator (7.56), so that the operation $O \mapsto O^E$ defines an *inner* derivation of $\mathcal{C}_E$. This extension of $\mathcal{C}$ may be realized explicitly by noting that the fermionic operators χ_0 and $\chi_0^\dagger$, obeying the Clifford algebra (7.59), force the vacuum state of the corresponding extended physical state space $\mathcal{H}_E$ to be two-fold degenerate. The two ground states are the $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum $|0\rangle$, with $\chi_0^\dagger|0\rangle = 0$, and its conjugate $Q^\dagger\,\chi_0|0\rangle$, where the operator $Q^\dagger$ is conjugate to the fermionic screening charge, i.e. $\{Q, Q^\dagger\} = 1$. From (7.59) and (7.61) it then follows that the field $\nabla_E(z)$ corresponds to the state $-\chi_0|0\rangle$. Furthermore, if $\mathcal{H}$ denotes the physical state space of the original WZW model, then the new space of states may be identified as $\mathcal{H}_E = \mathcal{H} \oplus \mathcal{H}$, with operators in $\mathcal{C}_E$ generically acting between the two copies of $\mathcal{H}$. As in section 6.1, these definitions immediately lead to logarithmic singularities in correlation functions of operators in $\mathcal{C}_E$.

Since $E(z)$ is a *fermionic* screening operator, the charge Q can serve as a differential in a complex. For $k = 2$, Q is the nilpotent BRST operator, and the space of states $\mathcal{H}$ can be defined as the cohomology of the screening operator acting in the larger space $\mathcal{H}_E$ [45]. For $k > 2$ we need to extend $\mathcal{H}_E$ further in order to guarantee nilpotency of Q. In any case, working with operators over the appropriate extended state space, from the above constructions it follows explicitly that

$$\begin{aligned} T^E(z)\, V^E_{j,m}(w) = & \sum_{n=1}^{\infty} \frac{1}{(z-w)^{n+2}} \sum_{l=1}^{n+1} (-1)^{n+l+1} \binom{n}{l-1} w^{n-l+1} \Big[E_{l-1}\,,\, V^E_{j,m}(w) \Big] + \\ & + \frac{\Delta_j}{(z-w)^2}\, V^E_{j,m}(w) + \frac{1}{(z-w)^2} \Big[Q^E\,,\, V^E_{j,m}(w) \Big] + \\ & + \frac{1}{z-w}\, \partial_w V^E_{j,m}(w) + \cdots\,, \end{aligned} \tag{7.62}$$

$$T^E(z) \Big[Q^E\,,\, V^E_{j,m}(w) \Big] = \frac{\Delta_j}{(z-w)^2} \Big[Q^E\,,\, V^E_{j,m}(w) \Big] + \frac{1}{z-w} \Big[Q^E\,,\, \partial_w V^E_{j,m}(w) \Big] + \cdots\,. \tag{7.63}$$

Thus the Jordan cells, of energy Δ_j, are spanned by $D_{j,m}(z) = V^E_{j,m}(z)$ and $C_{j,m}(z) = [Q^E, V^E_{j,m}(z)] \neq 0$ (note that Q^E is *not* a screening operator in the deformed chiral symmetry algebra).

To understand the appearence of rank 2 Jordan blocks in this context more clearly, let us consider the very special case of level $k = 2$. In the associated Virasoro representation provided by the Sugarawa construction, the decoupled bosonic sector of the Kac-Moody algebra can then be naturally embedded inside a topologically twisted $N = 2$ superconformal algebra [35] such that the screening operator $E(z)$ becomes one of the fermionic supercurrents. To see this, we note that the $N = 2$ algebra of central charge $c = 1$ has three unitary irreducible representations $\mathcal{H}_\ell$, $\ell = 0, 1, 2$, each of which decomposes as

$$\mathcal{H}_\ell = \bigoplus_{n=-\infty}^{\infty} \mathcal{F}_\ell^n \tag{7.64}$$

into Feigin-Fuks (free field Fock space) representations $\mathcal{F}_\ell^n$ of the $c = 0$ Virasoro algebra [40]. This is precisely the central charge of the decoupled bosonic WZW model at $k = 2$. The $N = 2$ algebra contains a U(1) current subalgebra, under which the Fock states of the module $\mathcal{F}_\ell^n$ all carry U(1) charge $n + \ell/3$. The vacuum state $|0\rangle_\ell$ of $\mathcal{H}_\ell$ satisfies the highest-weight conditions $E_n|0\rangle_\ell = 0 \ \ \forall n \geq \ell$ and $Q|0\rangle_0 = 0$. The screening operator Q, of U(1) charge -1, then provides a proper BRST differential making each $\mathcal{H}_\ell$ into an entire Felder complex of Virasoro modules over $\mathcal{C}$,

$$\dots \ \stackrel{Q}{\longrightarrow} \ \mathcal{F}_\ell^n \ \stackrel{Q}{\longrightarrow} \ \mathcal{F}_\ell^{n-1} \ \stackrel{Q}{\longrightarrow} \ \dots\,, \qquad Q^2 = 0\,. \tag{7.65}$$

The corresponding irreducible Virasoro representations, in the Fock spaces of the free scalar field which bosonizes the U(1) current of the $N = 2$ algebra, are then furnished through the BRST cohomology $\ker_{\mathcal{F}_\ell^n} Q / \mathrm{im}_{\mathcal{F}_\ell^{n+1}} Q$ [45].

Using this embedding, we now define the extension $\mathcal{C}_E$ of the chiral algebra $\mathcal{C}$ to be the extended $N = 2$ vertex operator algebra, constructed as explained above. Corresponding to $\mathcal{C}_E$ there are then three representations $\mathcal{L}_\ell$, $\ell = 0, 1, 2$. Because of (7.59), each $\mathcal{L}_\ell$ is an indecomposable extension of $\mathcal{H}_\ell$ by itself, i.e. there is a short exact sequence

$$0 \ \longrightarrow \ \mathcal{H}_\ell \ \longrightarrow \ \mathcal{L}_\ell \ \longrightarrow \ \mathcal{H}_\ell \ \longrightarrow \ 0\,. \tag{7.66}$$

As Virasoro representations, the decomposition (7.64), (7.65), along with (7.63), implies that (7.66) is a direct sum of sequences

$$0 \ \longrightarrow \ \mathcal{F}_\ell^n \ \longrightarrow \ \mathcal{L}_\ell^n \ \longrightarrow \ \mathcal{F}_\ell^{n+1} \ \longrightarrow \ 0\,, \qquad n \in \mathbb{Z}\,, \tag{7.67}$$

where $\mathcal{L}_\ell = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_\ell^n$ [13]. The non-diagonalizable Virasoro modules $\mathcal{L}_\ell^n$ carry definite U(1) charge $n \in \mathbb{Z}$, and they are generated by the original Kac-Moody currents of the $\mathrm{SU}(2)_2$ WZW model along with the extra zero modes χ_0, $\chi_0^\dagger$. In terms of the operator product expansions (7.62), (7.63), the C operators generate the sub-modules $\mathcal{F}_\ell^n \subset \mathcal{L}_\ell^n$, while the descendants of the field $\nabla_E(z)$, by which $\mathcal{C}$ is extended, live in the quotient space $\mathcal{L}_\ell^n / \mathcal{F}_\ell^n \cong \mathcal{F}_\ell^{n+1}$. This illustrates explicitly how the hamiltonian operator

$$L_0^E = L_0 + \chi_0^\dagger\, Q \tag{7.68}$$

for $k = 2$ acquires rank 2 Jordan blocks. For $k > 2$, we have not been able to find such indecomposable representations of the current algebra explicitly.

Finally, we relate the present Jordan block structure to the logarithmic pair (7.44), (7.45). For this, we write the fermionic screening operator (7.36) as

$$E(z) = \left(\partial_z\phi^2(z) + \sqrt{\frac{2}{k}}\,\partial_z\phi^4(z)\right) \ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\phi^1(z)}\,. \tag{7.69}$$

The deformation operator satisfying (7.55) is then given explicitly in the Coulomb gas representation by

$$\nabla_E(z) = \chi_0 + \left(\phi^2(z) + \sqrt{\frac{2}{k}}\,\phi^4(z)\right)\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\phi^1(z)} -$$
$$- \mathrm{i}\,\sqrt{\frac{2}{k}}\int^z \mathrm{d}w\ \left(\phi^2(w) + \sqrt{\frac{2}{k}}\,\phi^4(w)\right)\,\partial_w\phi^1(w)\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,\phi^1(w)}\,. \tag{7.70}$$

As usual, to neutralize the background charge of the ϕ^1 field, we introduce the vertex operator dual to (7.40) at $m = 0$,

$$\tilde{V}_{j,0}(z) = \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,(j+1)\,\phi^1(z)}\,. \tag{7.71}$$

Using (7.70) and the adjoint $\tilde{V}^\dagger_{j,0}(z)$ of (7.71), we now define the field

$$V^E_{j,\epsilon}(z) = \nabla_E(z)\cdot\left(z^{2\Delta_{j+1}}\,\tilde{V}_{j,0}(z^{-1})\right), \tag{7.72}$$

with ϵ defined by (5.7). Among other terms, the field $\phi^2(z)\ \mathrm{e}^{\,\mathrm{i}\,\sqrt{2/k}\,(-j\,\phi^1(z)+\epsilon\,\phi^2(z))}$ is present in the operator product (7.72), *provided* that the constraint (7.43) is satisfied. Up to normalization, this is just the logarithmic operator (7.45). Similarly, it is straightforward to check that $[Q^E, V^E_{j,\epsilon}(z)]$, with the constraint (7.43), contains the C operator (7.44).

The other operators arising in (7.62) and (7.72) may be attributed to higher-order terms in the deformation parameter $\epsilon \to 0^+$, as they are absent in the logarithmic conformal algebra of the operators (7.44), (7.45) to leading order in ϵ. While the Jordan blocks (7.62), (7.63) arise for generic values of j, m and k, it is only with the constraint (7.43) that the deformation leaves an isomorphic superconformal algebra and thus naturally explains the appearence of logarithmic operators in fusion relations within the original supersymmetric WZW model. In particular, it is only for $k = 2$ that we have been able to explicitly demonstrate both the appearence of the indecomposable representations and of the logarithmic operators through fusion relations. The analogy between the logarithmic pair (7.44), (7.45) and that of the recoil problem of section 5 is very suggestive of the fact that logarithmic operators are generally hidden within (subtle) marginal deformations of the original conformal algebra [13]. This feature is further supported by the similarity between (7.44), (7.45) and the logarithmic operators which arise in gravitationally dressed conformal field theories [5, 24].

7.5 Spin fields

The Coulomb gas representation, and the ensuing decoupling of fermions, allows the appropriate definition of the spin operator for the Ramond sector of the model. We seek an operator that introduces cuts in the free fermion fields $\psi^a(z)$, $a = 1, 2, 3$. For the fields $\psi^1(z)$ and $\psi^2(z)$ this is straightforward to construct using the technique of the previous sections. Namely, from the bosonization formula (7.25) it follows immediately that the ψ^1 and ψ^2 fields are twisted by the standard spin operator $\sqrt{2}\,\cos(\phi^4(z)/2)$ of dimension 1/8.

For the last unpaired Majorana fermion field $\psi^3(z)$, however, we cannot use bosonization and must treat it as a parafermion field, i.e. we must instead appeal to the standard, non-local oscillator construction of its spin field [46]. For this, we introduce the positive and negative frequency mode expansions of the operator $\psi^3(z) = \psi^3_{(\lambda)}(z)_+ + \psi^3_{(\lambda)}(z)_-$ in the Neveu-Schwarz ($\lambda = 0$) and Ramond ($\lambda = 1$) sectors by

$$\begin{aligned} \psi^3_{(\lambda)}(z)_+ &= \frac{1}{\sqrt{2}} \sum_{n<0} \psi^3_{n+(1-\lambda)/2}\; z^{-n-1+\lambda/2}\,, \\ \psi^3_{(\lambda)}(z)_- &= \frac{1}{\sqrt{2}} \sum_{n\geq 0} \psi^3_{n+(1-\lambda)/2}\; z^{-n-1+\lambda/2}\,, \end{aligned} \tag{7.73}$$

with $(\psi^3_r)^\dagger = \psi^3_{-r}$ and

$$\left\{\psi^3_r\,,\,\psi^3_s\right\} = \delta_{r+s,0}\,. \tag{7.74}$$

The ground states of the NS and R sectors obey the highest-weight conditions

$$\psi^3_{n+1/2}|0\rangle = \psi^3_{n+1}|\Delta\rangle_{\rm R} = 0 \qquad \forall n\geq 0 \tag{7.75}$$

with $\Delta = \hat c/16$. The $\mathbb{Z}_2$ twist field for $\psi^3(z)$ is then given by

$$\Pi(z) = {\rm e}^{\,z\,L^{\rm R}_{-1}}\,\langle 0|\,{\rm e}^{\,H(z)}\;\;{\rm e}^{\,I(z)}|\Delta\rangle_{\rm R}\,, \tag{7.76}$$

where

$$L^{\rm R}_{-1} = -\frac{1}{8}\sum_{n=-\infty}^{\infty}(2n+1)\,\psi^3_n\,\psi^3_{-n-1} \tag{7.77}$$

is the Virasoro translation generator acting in the Ramond sector of the Fock space defined by (7.73)–(7.75). The field

$$H(z) = \oint\limits_{w=z}\frac{{\rm d}w}{2\pi\,{\rm i}}\;\oint\limits_{w'=w}\frac{{\rm d}w'}{2\pi\,{\rm i}}\;\frac{1}{w-w'}\left(1-\sqrt{\frac{z-w}{z-w'}}\;\right)\psi^3_{(0)}(w)\,\psi^3_{(0)}(w') \tag{7.78}$$

is quadratic in the NS oscillators of $\psi^3(z)$, while the field

$$I(z) = \oint\limits_{w=z}\frac{{\rm d}w}{2\pi\,{\rm i}}\;\frac{w-z+1}{\sqrt{w-z}}\,\psi^3_{(0)}(w-z)_+\,\psi^3_{(1)}(w)_- \tag{7.79}$$

is bilinear in the NS and R oscillators.

The coefficients of the oscillator expansions of (7.78) and (7.79) are complicated functions of z [46]. In this representation, zero modes do not appear (in contrast to the bosonized representations), and the different vacua are described instead by the different states $|0\rangle$ and $|\frac{\hat c}{16}\rangle_{\rm R}$. Because the fermionic currents in this case coincide with the fermion fields themselves (see (7.31)), the twisting of the current algebra by the spin field $\Pi(z)$ is determined by the R sector operator product expansion

$$\psi^3(z)\,\Pi(w) = \frac{1}{\sqrt{z-w}}\,\pi(w) + \cdots\,, \tag{7.80}$$

where $\pi(z)$ is the corresponding excited twist field. The operator (7.76) has scaling dimension 1/16 [46].

The desired total spin field for the set of free fermion fields $\psi^a(z)$, $a = 1, 2, 3$ is therefore given by

$$\Sigma(z) = \sqrt{2}\,\cos\frac{\phi^4(z)}{2} \otimes \Pi(z)\,, \tag{7.81}$$

and it has dimension 3/16. Finally, the excited logarithmic spin fields defined by (2.22) may be computed explicitly by using (7.25), (7.52), (7.53) and (7.81) to get

$$\begin{aligned}
\widetilde{\Sigma}_{C_\epsilon}(z) &= 2\,\mathrm{i}\,\sqrt{\frac{1}{j(j+1)}}\,C_\epsilon(z)\otimes\left(-\sqrt{2}\,j\,\sin\frac{\phi^4(z)}{2}\otimes\Pi(z) + \mathrm{i}\,\epsilon\,\Sigma(z)\right)\,,\\
\widetilde{\Sigma}_{D_\epsilon}(z) &= \sqrt{j(j+1)}\,\left(D_\epsilon(z) + \frac{1}{\epsilon}\,C_\epsilon(z)\right)\otimes\Sigma(z)
\end{aligned} \tag{7.82}$$

in the limit $\epsilon \to 0^+$, and they have dimension 19/16. With these explicit representations, it is now straightforward to verify explicitly the relations of the R sector logarithmic superconformal algebra derived earlier. Let us note that at level $k = 2$, whereby the supersymmetric $su(2)_2$ current algebra is generated by a single scalar superfield, we may take $\phi^4(z) = 0$ in (7.81). Then the spin field $\Sigma(z)$ has dimension 1/16, and hence it creates the supersymmetric R sector ground state $|\hat{c}/16\rangle_{\mathrm{R}}$ at $k = 2$. On the other hand, for $k > 2$ there are three scalar superfield generators and $\Sigma(z)$ has dimension 3/16. Therefore, in the WZW models with $k > 2$, one has $\Sigma^-_{3/16}(z) \neq 0$ and the Ramond supersymmetry is *broken*. As indicated earlier, features such as this single out the value $k = 2$ as a very special point, and indeed it provides the lowest-lying marginal logarithmic deformation of section 7.3. We will see another very special feature of this level in the next subsection.

7.6 Extra $c = -2$ sectors

It has been suggested [23, 24] that the occurence of logarithmic behaviour in the bosonic $\mathrm{SU}(2)_0$ WZW model is due to the presence of an underlying $c = -2$ sector. We will now explore this possibility within the present context. Let us concentrate on the field $\phi^1(z)$ in the free field representation. From (7.22) its energy-momentum tensor is

$$T_{\phi^1}(z) = -\frac{1}{2}\,\partial_z\phi^1(z)\,\partial_z\phi^1(z) + \frac{\mathrm{i}}{\sqrt{2k}}\,\partial_z^2\phi^1(z)\,, \tag{7.83}$$

and hence it has central charge

$$c_{\phi^1} = 1 - \frac{6}{k}\,. \tag{7.84}$$

It follows that at $k = 2$, the Coulomb gas field $\phi^1(z)$ represents a bosonized symplectic fermion system with $c = c_{\phi^1} = -2$. We shall examine how this ϕ^1 sector of the field theory is related to the logarithmic structures that we have unveiled above. As we will see, the extra $c = -2$ sector which arises is responsible for the isomorphism between the deformed algebra of fields and the original superconformal algebra.

Explicitly, the bosonization formulas at $k=2$ are given by

$$\begin{aligned}
\eta(z) &= \frac{1}{\sqrt{2}}\,V_{1,0}(z) &= \frac{1}{\sqrt{2}}\ \mathrm{e}^{\,-\,\mathrm{i}\,\phi^1(z)}\,, \\
\xi(z) &= \frac{1}{\sqrt{2}}\,\tilde{V}_{0,0}(z) &= \frac{1}{\sqrt{2}}\ \mathrm{e}^{\,\mathrm{i}\,\phi^1(z)}\,, \\
\eta(z)\,\xi(z) &= -\frac{\mathrm{i}}{2}\,\partial_z\phi^1(z)\,. &
\end{aligned} \tag{7.85}$$

The fermionic $c=-2$ ghost system (η,ξ) has weight $(1,0)$ and non-vanishing two-point functions (6.23). Note that its bosonization is different from that of section 6.2, because now all modes of the field $\xi(z)$ will contribute. The energy-momentum tensor (7.83) for the ϕ^1 sector can then be written as

$$T_{\phi^1}(z) = -\eta(z)\,\partial_z\xi(z)\,, \tag{7.86}$$

and the logarithmic operators of section 7.3 may be expressed as

$$\begin{aligned}
C_\epsilon(z) &= -\sqrt{2}\,\mathrm{i}\,\epsilon\,\eta(z)\otimes\mathrm{e}^{\,\mathrm{i}\,\epsilon\,\phi^2(z)}\,, \\
D_\epsilon(z) &= -\sqrt{2}\,\eta(z)\otimes\phi^2(z)\ \mathrm{e}^{\,\mathrm{i}\,\epsilon\,\phi^2(z)}\,.
\end{aligned} \tag{7.87}$$

From (7.87) it is clear that at $k=2$ the logarithmic structure is *not* due to the presence of this underlying $c=-2$ sector in the theory, because the logarithmic behaviour of (7.87) arises from the ϕ^2 sector instead. Indeed, in the setting of section 7.3 the field $\phi^1(z)$ acts as a Liouville dressing for the operators constructed from the field $\phi^2(z)$. Thus the ϕ^1 sector in (7.87) ensures that the deformations are marginal and hence that the deformed algebra is conformal. This is even more apparent upon examination of the screening operator for the WZW model, which in this case yields directly the purely fermionic BRST operator Q that can be written in terms of the $c=-2$ ghost fields as

$$E(z) = \xi(z)\otimes\Big(\partial_z\phi^2(z)+\partial_z\phi^4(z)\Big)\,. \tag{7.88}$$

Deforming the current algebra in the manner described in section 7.4 using the spin 0 ghost field in (7.88) will not produce the logarithmic deformation (7.87) that is determined instead by the spin 1 ghost field.

The main reason for this discrepency here lies in the fact that the field $\xi(z)$ is *not* a screening operator in the $c=-2$ model. Let us introduce the mode expansions of the fermionic ghost fields,

$$\begin{aligned}
\eta(z) &= \eta_0 - \sum_{n\neq0}\eta_n\ z^{-n-1}\,, \\
\xi(z) &= \xi_0 + \sum_{n\neq0}\frac{\xi_n}{n}\ z^{-n}\,,
\end{aligned} \tag{7.89}$$

with the non-vanishing anticommutation relations

$$\begin{aligned}
\{\eta_n,\xi_m\} &= n\,\delta_{n+m,0}\,, \qquad n\neq0\,, \\
\{\eta_0,\xi_0\} &= 1\,,
\end{aligned} \tag{7.90}$$

and the vacuum state annihilation conditions $\eta_n|0\rangle = \xi_{n+1}|0\rangle = 0 \quad \forall n \geq 0$. From the bosonization formulas (7.85) it follows that a fermionic screening operator for the $c = -2$ model is given by the zero mode of the η field,

$$\eta_0 = \oint_{z=0} \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\ \eta(z)\,. \tag{7.91}$$

The presence of the extra screening operator $\eta(z)$ in fact opens the door for the construction of yet *another* logarithmic deformation $\mathcal{C}_\eta$ of the current algebra $\mathcal{C}$, with the same properties as that constructed in section 7.4 and in parallel with the symplectic fermion model of section 6.1 [13].

For this, we note that the deformation field of $\mathcal{C}_\eta$ is the operator

$$\nabla_\eta(z) = \chi_0 + \eta_0\ \ln z + \sum_{n\neq 0} \frac{\eta_n}{n}\ z^{-n}\,, \tag{7.92}$$

and the corresponding deformation of the ξ field is given by

$$\xi^\eta(z) = \xi_0 + \chi_0^\dagger\ \ln z + \sum_{n\neq 0} \frac{\xi_n}{n}\ z^{-n}\,. \tag{7.93}$$

The extra zero mode $\chi_0^\dagger$ generated for the field $\xi^\eta(z)$, along with the anticommutation relations (7.59) and (7.90), leads to the desired logarithmic correlation

$$\nabla_\eta(z)\,\xi^\eta(w) = \ln(z-w) + \cdots\,. \tag{7.94}$$

Moreover, the deformed energy-momentum tensor is given by

$$T^\eta_{\phi^1}(z) = -\eta(z)\,\partial_z\xi(z) + \frac{\chi_0^\dagger}{z}\,\eta(z) = -\eta(z)\,\partial_z\xi^\eta(z)\,, \tag{7.95}$$

and its form invariance in terms of the transformed fields is a direct consequence of the fact [13] that the operator products are preserved by the deformation. These are of course just the standard relations of the symplectic fermion field theory that we studied in section 6.1, with the identifications $(\nabla_\eta\,,\,\xi^\eta) = (-\chi^-\,,\,\chi^+)$.

The indecomposable module extensions in $\mathcal{C}_\eta$ are determined by the Felder complex $\eta_0 : \mathcal{F}^n \to \mathcal{F}^{n+1}$ with Feigin-Fuks representations $\mathcal{F}^n$ consisting of Fock space states of fixed (η,ξ) ghost number $n \in \mathbb{Z}$ [13]. In the mapping from the $c = 0$ model to the $N = 2$ superconformal field theory that we discussed in section 7.4, this is the same U(1) current that is used to grade the Felder complex (7.65) [21]. However, we stress again that the screenings in the two sectors are completely different, and hence so are the logarithmic deformations. For the symplectic fermion, the logarithmic pair in the ϕ^1 sector for $n = 0$ is given by $(C\,,\,D) = (I\,,\,\nabla_\eta\,\xi^\eta)$, and it generates a Jordan cell spanned by the states $(|0\rangle\,,\,\xi_0\,\chi_0|0\rangle)$. Note that this Jordan block has spin 0, and so the logarithmic deformation does not preserve the conformal algebra. This is in contrast to the spin 1 blocks generated by (7.87) that arise from primary field fusion relations and have U(1) ghost charge $n = 1$.

It follows from this analysis that there is another, independent logarithmic sector of the supersymmetric $\mathrm{SU}(2)_2$ WZW model. It is not the same as that which comes about from logarithmic singularities in the WZW conformal blocks. The full current algebra has much more symmetry and a somewhat richer logarithmic structure. Nevertheless, several striking similarities between the triplet model and the WZW model seem to arise due to this underlying ghost system [23, 24]. Of course, this analysis breaks down at higher levels $k > 2$. On the other hand, for generic values of the level $k \in \mathbb{Z}_+$, there exist alternative (but equivalent) free field representations of the Kac-Moody currents which make manifest an underlying $c = -2$ structure. The bosonic currents can be represented by either the three scalar fields $\varphi^a(z)$ studied above, or they can be "debosonized" into ghost fields (β, γ) [13, 23, 28]. These ghost fields constitute a first order *bosonic* system and a current such that the $\mathrm{SU}(2)_{k-2}$ currents become those of the Wakimoto construction [22]. The ghost system (β, γ) may then be bosonized in terms of a symplectic fermion system (η, ξ), as in section 6.2, which can be used to construct a logarithmic conformal algebra by deforming the fields as above. In this way one can obtain logarithmic deformations, at least in a certain sector of the quantum field theory, for generic values of $k \in \mathbb{Z}_+$ in supersymmetric $\mathrm{SU}(2)_k$ WZW models. However, it should be stressed again that such deformations only lead to isomorphic superconformal algebras when the highest-weight constraint (7.43) is obeyed.

7.7 Supersymmetric $sl(2, \mathbb{R})_k$ current algebras

For completeness we shall now briefly describe the situation in the case of non-compact current algebras, which can be treated in a similar manner. Formally, one obtains the non-compact case from the compact one essentially by flipping the sign $k \mapsto -k$ of the level number $k \in \mathbb{Z}_+$, as a sort of analytic continuation of the field theory [47]. In supersymmetric $sl(2, \mathbb{R})_k$ WZW models one can again demonstrate the decoupling of the supersymmetric fermionic partners, and hence an equivalence of the bosonic sector at level $k-2$. It therefore suffices for our purposes again to concentrate on bosonic $sl(2, \mathbb{R})_{k-2}$ affine algebras.

The bosonic currents in this case, for an arbitrary level $k \in \mathbb{Z}_+$, can be represented by means of free scalar fields. A specific representation may be given in which the two currents $J^+(z)$ and $J^0(z)$ are represented in terms of two scalar fields $\phi(z)$ and $\varphi(z)$, while the third current $J^-(z)$ depends explicitly on the energy-momentum tensor $T(z)$ [13]. The field $T(z)$ can itself be bosonised by means of a scalar current $\partial_z u(z)$, and the entire system debosonized to ghost fields (β, γ) plus a current in such a way that the Wakimoto representation [22] for the currents arises. The logarithmic deformations then appear through a fermionic screening charge as before [13], and are given by the zero mode η_0 of the $c = -2$ system (η, ξ) that is used in the bosonization of the ghost system (β, γ). The crucial difference from the previous analysis is that now the operator η_0 *coincides* exactly with the screening operator of the $sl(2, \mathbb{R})_{k-2}$ current algebra for any $k \in \mathbb{Z}_+$. Thus the logarithmic behaviour in this class of WZW models is indeed due entirely to symplectic fermions and occurs for generic values of the level coefficient. In particular, the bosonic part of the current algebra has central charge $c = -2$ at $k = 2$, and hence logarithmic behaviour arises directly at this special level number due to the presence of an underlying $c = -2$ sector of the quantum field theory.

For completeness, in order to compare with the corresponding expression (7.36) for the compact current algebras, we note that a fermionic screening operator in the present case is given by

$$Q' = \oint_{z=0} \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\ \mathrm{e}^{\sqrt{(k-2)/2}\,\phi(z)}\,\mathcal{V}_{1,2}(z)\,, \tag{7.96}$$

where $\mathcal{V}_{r,s}(z)$ are Virasoro primary fields of conformal dimension

$$\Delta_{r,s} = \frac{r^2-1}{4k} + \frac{s^2-1}{4}\,k + \frac{1-rs}{2}\,. \tag{7.97}$$

Notice that the exponential in (7.96) is real-valued and amounts to the analytic continuation $k-2 \mapsto -(k-2)$ in the passage from the compact $su(2)$ current algebra to the non-compact $sl(2,\mathbb{R})$ case. The $\mathrm{SL}(2,\mathbb{R})$ primary fields are given by [13]

$$V_{r,s}(z) = \mathrm{e}^{\,j(r,s)\,\sqrt{(k-2)/2}\,(\varphi(z)-\phi(z))}\,\mathcal{V}_{r,s}(z)\,, \tag{7.98}$$

where

$$j(r,s) = \frac{r-1}{2} - \frac{s-1}{2}\,k\,. \tag{7.99}$$

Using these primary states one can now proceed in an analogous way to the compact case above and demonstrate explicitly the appearance of logarithmic deformations for generic values of the level k. The indecomposable extensions of the corresponding highest-weight representations $[V_{r,s}]$ are constructed explicitly in [13], as are the corresponding states which generate the Jordan cells. The appearence of logarithmic operators through fusion relations of spin $j = -1/2$ primary fields is described in [25]. These models are essentially equivalent to Liouville theory, which at $c = 1$ is known to contain logarithmic operators [5]. As indicated earlier, the logarithmic operators in these theories at generic k [24, 25] are strikingly similar to the ones we have found in section 7.3 above. These non-compact current algebras also describe string propagation in an AdS_3 background [48].

7.8 Supersymmetric coset models

Finally, let us mention some brief points about coset constructions, although we will not deal with these models explicitly here. The analysis of such models is complicated by the fact that the decoupling of fermionic partners is not complete. The equivalence of coset models over G/H with a gauged $N = 1$ supersymmetric WZW model, in which a diagonal subgroup $H \subset G$ is gauged, has been analysed in [49]. There it is demonstrated, by means of appropriate fermionic currents as above, that the fermion fields in the adjoint representation of the subgroup $H \subset G$ decouple. One is then left with a theory of coset fermion fields living in G/H which are *gauged*, and also an ordinary WZW model in which the diagonal H symmetry is gauged. In the limit $H = G$ there are no coset fermion fields and the gauged supersymmetric WZW model reduces to an ordinary WZW model in the sense of complete decoupling of fermion fields, as we have seen above. The detailed analysis of such models represents an interesting arena in which to search for logarithmic operators. They are also relevant to the standard coset constructions of $N = 2$ superconformal field theories. They might therefore lead to logarithmic $N = 2$ superconformal algebras and, in the string context, there may even be the possibility of breaking $N = 2$ worldsheet supersymmetry, in an analogous way that $N = 1$ supersymmetry was broken in the $k > 2$ WZW models (see section 7.5).

These theories, combined with those of the previous subsection, also represent interesting physical situations where logarithmic behaviour arises. For example, logarithmic operators may arise in two-dimensional black holes which can be described as exact gauged WZW conformal field theories over the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ [47]. The logarithmic behaviour in this case may be attributed to an exactly marginal deformation of the black hole background connected with the W_∞-symmetry of the target space [6]. Again, the logarithmic operators which arise in this case [24] are very similar to those of section 7.3. More recently, logarithmic behaviour has been discovered in the coset model $\mathrm{SU}(2)_k/\,\mathrm{U}(1) \times \mathrm{U}(1)_{-k}$ [50] (and also in more general cosets $G_k/H \times \mathrm{U}(1)_{-k}$ [51]), which admits a spacetime interpretation as an exact three-dimensional plane wave solution of supergravity in a correlated Penrose limit which involves taking $k \to \infty$. The main ingredients of the construction of logarithmic operators in this case, namely the usage of a small deformation parameter $\epsilon = 1/k$ and a free time-like boson field, are exactly the same as those used in section 7.3.

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